钱学森力学手稿

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XI'AN JIAOTONG UNIVERSITY PRESS

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出版前言

2011年12月11日是西安交通大学杰出校友钱学森先生的百年诞辰。为缅怀钱学森学长，学习他的科学思想和卓越风范，展示其丰功伟绩和人格魅力，西安交通大学举办了“纪念钱学森诞辰100周年”系列活动：作为制片方之一，参与西部电影集团摄制传记故事片《钱学森》；与中央电视台合作，出品纪录片《实验班的故事——沿着钱学森走过的路》；扩建钱学森生平业绩展馆，向校内外开放；举办钱学森科学与教育思想研讨会；出版发行《钱学森力学手稿》、《钱学森年谱（初编）》、《钱学森第六次产业革命思想探微丛书》等。

钱学森先生在美国深造和工作期间留下大量珍贵手稿，这些手稿真实展示了钱学森先生博大精深的学识、开拓求实的精神和严谨奋进的作风，是钱老勇攀科学高峰和严谨治学的集中体现。这里，我们将部分原稿整理汇集成册，出版《钱学森力学手稿》，作为钱老百年诞辰的献礼。

《钱学森力学手稿》共10卷，包含两部分内容。第一部分是草稿，包括扁壳、球壳和圆柱壳屈曲分析的公式推导和数值演算。在研究圆柱壳轴压屈曲问题时，为了求得圆柱壳体的临界压力，在有关的五百多页草稿中，对多达二十多种可能的屈曲模

态逐一进行公式推演和数值计算，最终才找到满意的并在论文中采用的屈曲模态。仔细观察草稿中的数据列表，每个数字有效位数都长达八位，在手摇机械式计算机作为主要计算工具的年代，这串串数字凝聚着多少现今难以想象的艰辛劳动。

第二部分是手稿，以航空航天工程为核心，涵盖空气动力学、固体力学、火箭技术、工程控制论和物理力学等领域的部分学术论文手稿、打印稿和讲义。

《钱学森力学手稿》是在西安交通大学校领导的大力支持下，由西安交通大学航天航空学院沈亚鹏教授整理完成。图书出版过程中得到了西安交通大学党委宣传部、校友关系发展部、图书馆、航天航空学院等的积极协助，在此深表感谢。

Preliminary Calculation of Circular Cylinder (Ⅲ)

New Calculation

for the circular region,

$$\frac{w_0}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left\{1-\left(\frac{r}{a}\right)^2\sin^2\theta\right\}$$

$$\frac{w}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left\{1-\left(\frac{r}{a}\right)^2\sin^2\theta - f_1\left[1-\left(\frac{r}{a}\right)^2\right]^2 - f_2\left[1-\left(\frac{r}{a}\right)^2\right]^6\right\}$$

$$\frac{1}{r}\frac{\partial w}{\partial r} = \frac{1}{R}\left\{-\sin^2\theta + 2f_1\left[1-\left(\frac{r}{a}\right)^2\right] + 6f_2\left[1-\left(\frac{r}{a}\right)^2\right]^5\right\}$$

$$\frac{1}{r}\frac{\partial w_0}{\partial r} = \frac{1}{R}\left\{-\sin^2\theta\right\}$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{1}{R}\left\{-\sin^2\theta + 2f_1\left[1-3\left(\frac{r}{a}\right)^2\right] + 6f_2\left[1-\left(\frac{r}{a}\right)^2\right]^4\left[1-11\left(\frac{r}{a}\right)^2\right]\right\}$$

$$\frac{\partial^2 w_0}{\partial r^2} = \frac{1}{R}\left\{-\sin^2\theta\right\}$$

$$\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} = \frac{1}{R}\left\{-\cos 2\theta\right\}$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\}$$

$$=\frac{1}{R^2}\Bigg[(1-\cos 2\theta)\left\{2f_1\left[1-2\left(\tfrac{r}{a}\right)^2\right]+6f_2\left[1-\left(\tfrac{r}{a}\right)^2\right]^4\left[1-6\left(\tfrac{r}{a}\right)^2\right]\right\}$$

$$-\left\{4f_1^2\left[1-\left(\tfrac{r}{a}\right)^2\right]\left[1-3\left(\tfrac{r}{a}\right)^2\right]+24f_1f_2\left[1-\left(\tfrac{r}{a}\right)^2\right]^5\left[1-7\left(\tfrac{r}{a}\right)^2\right]+36f_2^2\left[1-\left(\tfrac{r}{a}\right)^2\right]^9\left[1-11\left(\tfrac{r}{a}\right)^2\right]\right\}$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial\theta^2}-\frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial\theta^2}\right\}=\frac{1}{R^2}\cos 2\theta\left\{2f_1\left[1-3\left(\tfrac{r}{a}\right)^2\right]+6f_2\left[1-\left(\tfrac{r}{a}\right)^2\right]^4\left[1-11\left(\tfrac{r}{a}\right)^2\right]\right\}$$

The terms for the particular integral is then, multiplied by R^2

$$2f_1\left[1-2\left(\tfrac{r}{a}\right)^2\right]+6f_2\left[1-\left(\tfrac{r}{a}\right)^2\right]^4\left[1-6\left(\tfrac{r}{a}\right)^2\right]-4f_1^2\left[1-\left(\tfrac{r}{a}\right)^2\right]\left[1-3\left(\tfrac{r}{a}\right)^2\right]$$

$$-24f_1f_2\left[1-\left(\tfrac{r}{a}\right)^2\right]^5\left[1-7\left(\tfrac{r}{a}\right)^2\right]-36f_2^2\left[1-\left(\tfrac{r}{a}\right)^2\right]^9\left[1-11\left(\tfrac{r}{a}\right)^2\right]$$

$$+\cos 2\theta\left\{-2f_1\left(\tfrac{r}{a}\right)^2-30f_2\left[1-\left(\tfrac{r}{a}\right)^2\right]^4\left(\tfrac{r}{a}\right)^2\right\}$$

1	$\left(\frac{r}{a}\right)^2$	$\left(\frac{r}{a}\right)^4$	$\left(\frac{r}{a}\right)^6$	$\left(\frac{r}{a}\right)^8$	$\left(\frac{r}{a}\right)^{10}$	$\left(\frac{r}{a}\right)^{12}$	$\left(\frac{r}{a}\right)^{14}$	$\left(\frac{r}{a}\right)^{16}$	$\left(\frac{r}{a}\right)^{18}$	$\left(\frac{r}{a}\right)^{20}$
$+2f_1$	$-4f_1$									
$+6f_2$	$-24f_2$ $-36f_2$	$+36f_2$ $+144f_2$	$-24f_2$ $-216f_2$	$+6f_2$ $+144f_2$	$-36f_2$					
$-4f_1^2$	$+16f_1^2$	$-12f_1^2$								
$-48f_1f_2$	$+240f_1f_2$ $+336f_1f_2$	$-480f_1f_2$ $-1680f_1f_2$	$+480f_1f_2$ $+3360f_1f_2$	$-240f_1f_2$ $-3360f_1f_2$	$+48f_1f_2$ $+1680f_1f_2$	$-336f_1f_2$	$\times$	$\frac{1}{2}$		
$-36f_2^2$	$+324f_2^2$ $+396f_2^2$	$-1296f_2^2$ $-3564f_2^2$	$+3024f_2^2$ $+14256f_2^2$	$-4536f_2^2$ $-33264f_2^2$	$+4536f_2^2$ $+49896f_2^2$	$-3024f_2^2$ $-49896f_2^2$	$+1296f_2^2$ $+33264f_2^2$	$-324f_2^2$ $-14256f_2^2$	$+36f_2^2$ $+3564f_2^2$	$-396f_2^2$
$\frac{1}{16\cdot 4}$	$\frac{1}{36\cdot 16}$	$\frac{1}{64\cdot 36}$	$\frac{1}{100\cdot 64}$	$\frac{1}{144\cdot 100}$	$\frac{1}{196\cdot 144}$	$\frac{1}{256\cdot 196}$	$\frac{1}{324\cdot 256}$	$\frac{1}{400\cdot 324}$	$\frac{1}{484\cdot 400}$	$\frac{1}{576\cdot 484}$
	$-2f_1$									
	$-30f_2$	$+120f_2$	$-180f_2$	$+120f_2$	$-30f_2$					
	$\frac{1}{32\cdot 12}$	$\frac{1}{60\cdot 32}$	$\frac{1}{96\cdot 60}$	$\frac{1}{140\cdot 96}$	$\frac{1}{192\cdot 140}$					

$$\frac{\Phi}{R^2} = E\left(\frac{a}{R}\right)^4\Big[\frac{1}{4}\gamma_0\left(\frac{r}{a}\right)^2 + \frac{1}{32}\left(f_1 + 3f_2 - 2f_1^2 - 12f_1f_2 - 18f_2^2\right)\left(\frac{r}{a}\right)^4 + \frac{1}{144}\left(-f_1 - 15f_2 + 4f_1^2 + 72f_1f_2 + 180f_2^2\right)\left(\frac{r}{a}\right)^6$$

$$+ \frac{1}{192}\left(15f_2 - f_1^2 - 90f_1f_2 - 405f_2^2\right)\left(\frac{r}{a}\right)^8 + \frac{1}{400}\left(-15f_2 + 120f_1f_2 + 1080f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$+ \frac{1}{96}\left(f_2 - 12f_1f_2 - 1252f_2^2\right)\left(\frac{r}{a}\right)^{12} + \frac{1}{784}\left(-f_2 + 24f_1f_2 + 1512f_2^2\right)\left(\frac{r}{a}\right)^{14}$$

$$+ \frac{1}{896}\left(-3f_1f_2 - 945f_2^2\right)\left(\frac{r}{a}\right)^{16} + \frac{5}{12}f_2^2\left(\frac{r}{a}\right)^{18} - \frac{9}{80}f_2^2\left(\frac{r}{a}\right)^{20} + \frac{9}{484}f_2^2\left(\frac{r}{a}\right)^{22} - \frac{1}{704}f_2^2\left(\frac{r}{a}\right)^{24}$$

$$+ \cos 2\theta\Big\{-\frac{1}{192}\left(f_1 + 15f_2\right)\left(\frac{r}{a}\right)^6 + \frac{1}{16}f_2\left(\frac{r}{a}\right)^8 - \frac{1}{32}f_2\left(\frac{r}{a}\right)^{10} + \frac{1}{112}f_2\left(\frac{r}{a}\right)^{12} - \frac{1}{896}f_2\left(\frac{r}{a}\right)^{14} + p_2\left(\frac{r}{a}\right)^2$$

$$+ h_2\left(\frac{r}{a}\right)^4\Big\}\Big]$$

$$\frac{1}{r}\frac{\partial\Phi}{\partial r} = E\left(\frac{a}{R}\right)^2\Big[\frac{1}{2}\gamma_0 + \frac{1}{8}A\left(\frac{r}{a}\right)^2 + \frac{1}{24}B\left(\frac{r}{a}\right)^4 + \frac{1}{24}C\left(\frac{r}{a}\right)^6 + \frac{1}{40}D\left(\frac{r}{a}\right)^8 + \frac{1}{8}E\left(\frac{r}{a}\right)^{10} + \frac{1}{56}F\left(\frac{r}{a}\right)^{12}$$

$$+ \frac{1}{56}G\left(\frac{r}{a}\right)^{14} + \frac{15}{2}H\left(\frac{r}{a}\right)^{16} - \frac{9}{4}I\left(\frac{r}{a}\right)^{18} + \frac{9}{22}J\left(\frac{r}{a}\right)^{20} - \frac{3}{88}K\left(\frac{r}{a}\right)^{22}$$

$$+ \cos 2\theta\Big\{-\frac{6}{192}\left(f_1 + 15f_2\right)\left(\frac{r}{a}\right)^4 + \frac{8}{16}f_2\left(\frac{r}{a}\right)^6 - \frac{10}{32}f_2\left(\frac{r}{a}\right)^8 + \frac{12}{112}f_2\left(\frac{r}{a}\right)^{10} - \frac{14}{896}f_2\left(\frac{r}{a}\right)^{12}$$

$$+ 2p_2 + 4h_2\left(\frac{r}{a}\right)^2\Big\}\Big]$$

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$$\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = E\left(\frac{a}{R}\right)^2 \cos 2\theta \left\{ \frac{4}{192}(f_1+15f_2)\left(\frac{r}{a}\right)^4 - \frac{4}{16} f_2 \left(\frac{r}{a}\right)^6 + \frac{4}{32} f_2\left(\frac{r}{a}\right)^8 - \frac{4}{112} f_2 \left(\frac{r}{a}\right)^{10} + \frac{4}{896} f_2\left(\frac{r}{a}\right)^{12} - 4p_2 - 4r_2\left(\frac{r}{a}\right)^2 \right\}$$

$$\widehat{rr} = E\left(\frac{a}{R}\right)^2 \Bigg[\frac{1}{2}q_0 + \frac{1}{8}A\left(\frac{r}{a}\right)^2 + \frac{1}{24}B\left(\frac{r}{a}\right)^4 + \frac{1}{24}C\left(\frac{r}{a}\right)^6 + \frac{1}{40}D\left(\frac{r}{a}\right)^8 + \frac{1}{8}E\left(\frac{r}{a}\right)^{10} + \frac{1}{56}F\left(\frac{r}{a}\right)^{12}$$

$$+ \frac{1}{56}G\left(\frac{r}{a}\right)^{14} + \frac{15}{2}H\left(\frac{r}{a}\right)^{16} - \frac{9}{4}I\left(\frac{r}{a}\right)^{18} + \frac{9}{22}J\left(\frac{r}{a}\right)^{20} - \frac{3}{88}K\left(\frac{r}{a}\right)^{22}$$

$$+ \cos 2\theta \left\{ -\frac{2}{192}(f_1+15f_2)\left(\frac{r}{a}\right)^4 + \frac{4}{16} f_2\left(\frac{r}{a}\right)^6 - \frac{6}{32} f_2\left(\frac{r}{a}\right)^8 + \frac{8}{112} f_2\left(\frac{r}{a}\right)^{10} - \frac{10}{896} f_2\left(\frac{r}{a}\right)^{12} - 2p_2 \right\} \Bigg]$$

$$\widehat{\theta\theta} = E\left(\frac{a}{R}\right)^2 \Bigg[\frac{1}{2}q_0 + \frac{3}{8}A\left(\frac{r}{a}\right)^2 + \frac{5}{24}B\left(\frac{r}{a}\right)^4 + \frac{7}{24}C\left(\frac{r}{a}\right)^6 + \frac{9}{40}D\left(\frac{r}{a}\right)^8 + \frac{11}{8}E\left(\frac{r}{a}\right)^{10} + \frac{13}{56}F\left(\frac{r}{a}\right)^{12}$$

$$+ \frac{15}{56}G\left(\frac{r}{a}\right)^{14} + \frac{15\times 17}{2}H\left(\frac{r}{a}\right)^{16} - \frac{9\times 19}{4}I\left(\frac{r}{a}\right)^{18} + \frac{9\times 21}{22}J\left(\frac{r}{a}\right)^{20} - \frac{3\times 23}{88}K\left(\frac{r}{a}\right)^{22}$$

$$+ \cos 2\theta \left\{ -\frac{30}{192}(f_1+15f_2)\left(\frac{r}{a}\right)^4 + \frac{56}{16} f_2\left(\frac{r}{a}\right)^6 - \frac{90}{32} f_2\left(\frac{r}{a}\right)^8 + \frac{132}{112} f_2\left(\frac{r}{a}\right)^{10} - \frac{182}{896} f_2\left(\frac{r}{a}\right)^{12} + 2p_2 + 12r_2\left(\frac{r}{a}\right)^2 \right\} \Bigg]$$

$$\widehat{r\theta} = E\left(\frac{a}{R}\right)^2 \sin 2\theta \left\{ -\frac{10}{192}(f_1+15f_2)\left(\frac{r}{a}\right)^4 + \frac{14}{16} f_2\left(\frac{r}{a}\right)^6 - \frac{18}{32} f_2\left(\frac{r}{a}\right)^8 + \frac{22}{112} f_2\left(\frac{r}{a}\right)^{10} - \frac{26}{896} f_2\left(\frac{r}{a}\right)^{12} + 2p_2 + 6r_2\left(\frac{r}{a}\right)^2 \right\}$$

$$\hat{n}-\nu\hat{\theta}=E\left(\frac{a}{R}\right)^2\left[\frac{1}{2}(1-\nu)q_0+\frac{(1-3\nu)}{8}A\left(\frac{r}{a}\right)^2+\frac{(1-5\nu)}{24}B\left(\frac{r}{a}\right)^4+\frac{(1-7\nu)}{24}C\left(\frac{r}{a}\right)^6+\frac{(1-9\nu)}{40}D\left(\frac{r}{a}\right)^8+\frac{(1-11\nu)}{18}E\left(\frac{r}{a}\right)^{10}\right.$$

$$+\frac{(1-13\nu)}{56}F\left(\frac{r}{a}\right)^{12}$$

$$+\frac{(1-15\nu)}{56}G\left(\frac{r}{a}\right)^{14}+\frac{15(1-17\nu)}{2}H\left(\frac{r}{a}\right)^{16}-\frac{9(1-19\nu)}{4}I\left(\frac{r}{a}\right)^{18}+\frac{9(1-21\nu)}{22}J\left(\frac{r}{a}\right)^{20}-\frac{3(1-23\nu)}{88}K\left(\frac{r}{a}\right)^{22}$$

$$+\cos 2\theta\left\{-\frac{(1-15\nu)}{96}(f_1+15f_2)\left(\frac{r}{a}\right)^4+\frac{(1-14\nu)}{4}f_2\left(\frac{r}{a}\right)^6-\frac{(3-45\nu)}{16}f_2\left(\frac{r}{a}\right)^8+\frac{(2-33\nu)}{28}f_2\left(\frac{r}{a}\right)^{10}-\frac{(5-91\nu)}{448}f_2\left(\frac{r}{a}\right)^{12}\right.$$

$$\left.\left.-2(1+\nu)p_2-12\nu r_2\left(\frac{r}{a}\right)^2\right\}\right]$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2-\frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2=\left(\frac{a}{R}\right)^2\left[-f_1\left[1-\left(\frac{r}{a}\right)^2\right]\left(\frac{r}{a}\right)^2-3f_2\left[1-\left(\frac{r}{a}\right)^2\right]^5\left(\frac{r}{a}\right)^2+2f_1^2\left[1-\left(\frac{r}{a}\right)^2\right]^2\left(\frac{r}{a}\right)^2+12f_1f_2\left[1-\left(\frac{r}{a}\right)^2\right]^6\left(\frac{r}{a}\right)^2\right.$$

$$\left.+18f_2^2\left[1-\left(\frac{r}{a}\right)^2\right]^{10}\left(\frac{r}{a}\right)^2+\cos 2\theta\left\{f_1\left[1-\left(\frac{r}{a}\right)^2\right]\left(\frac{r}{a}\right)^2+3f_2\left[1-\left(\frac{r}{a}\right)^2\right]^5\left(\frac{r}{a}\right)^2\right\}\right]$$

$$=\left(\frac{a}{R}\right)^2\left[-A\left(\frac{r}{a}\right)^2-B\left(\frac{r}{a}\right)^4-2C\left(\frac{r}{a}\right)^6-9D\left(\frac{r}{a}\right)^8-15E\left(\frac{r}{a}\right)^{10}-3F\left(\frac{r}{a}\right)^{12}-4G\left(\frac{r}{a}\right)^{14}-2160H\left(\frac{r}{a}\right)^{16}\right.$$

$$+810\,I\left(\frac{r}{a}\right)^{18}-180J\left(\frac{r}{a}\right)^{20}+18K\left(\frac{r}{a}\right)^{22}$$

$$\left.+\cos 2\theta\left\{(f_1+3f_2)\left(\frac{r}{a}\right)^2-(f_1+15f_2)\left(\frac{r}{a}\right)^4+30f_2\left(\frac{r}{a}\right)^6-30f_2\left(\frac{r}{a}\right)^8+15f_2\left(\frac{r}{a}\right)^{10}-3f_2\left(\frac{r}{a}\right)^{12}\right\}\right]$$

$\left(\frac{r}{a}\right)^2$	$\left(\frac{r}{a}\right)^4$	$\left(\frac{r}{a}\right)^6$	$\left(\frac{r}{a}\right)^8$	$\left(\frac{r}{a}\right)^{10}$	$\left(\frac{r}{a}\right)^{12}$	$\left(\frac{r}{a}\right)^{14}$	$\left(\frac{r}{a}\right)^{16}$	$\left(\frac{r}{a}\right)^{18}$	$\left(\frac{r}{a}\right)^{20}$	$\left(\frac{r}{a}\right)^{22}$
$-f_1$	$+f_1$									
$-3f_2$	$+15f_2$	$-30f_2$	$+30f_2$	$-15f_2$	$+3f_2$					
$+2f_1^2$	$-4f_1^2$	$+2f_1^2$								
$+12f_1f_2$	$-72f_1f_2$	$+180f_1f_2$	$-240f_1f_2$	$+180f_1f_2$	$-72f_1f_2$	$+12f_1f_2$				
$+18f_2^2$	$-180f_2^2$	$+810f_2^2$	$-2160f_2^2$	$+3780f_1^2$	$-4536f_1f_2$	$+3780f_2^2$	$-2160f_2^2$	$+810f_2^2$	$-180f_2^2$	$+18f_2^2$
$+f_1$	$-f_1$									
$+3f_2$	$-15f_2$	$+30f_2$	$-30f_2$	$+15f_2$	$-3f_2$					

$$\frac{\partial u}{\partial r} = \left(\frac{q}{R}\right)^2 \Big[\frac{1}{2}(1-\nu)q_0 + \frac{3(3-\nu)}{8}A\left(\frac{r}{a}\right)^2 + \frac{5(5-\nu)}{24}B\left(\frac{r}{a}\right)^4 + \frac{7(7-\nu)}{24}C\left(\frac{r}{a}\right)^6 + \frac{9(9-\nu)}{40}D\left(\frac{r}{a}\right)^8$$

$$+ \frac{11(11-\nu)}{8}E\left(\frac{r}{a}\right)^{10} + \frac{13(13-\nu)}{56}F\left(\frac{r}{a}\right)^{12} + \frac{15(15-\nu)}{56}G\left(\frac{r}{a}\right)^{14} + \frac{15\times 17(17-\nu)}{2}H\left(\frac{r}{a}\right)^{16} - \frac{9\times 19(19-\nu)}{4}I\left(\frac{r}{a}\right)^{18}$$

$$+ \frac{9\times 21(21-\nu)}{22}J\left(\frac{r}{a}\right)^{20} - \frac{3\times 23(23-\nu)}{88}K\left(\frac{r}{a}\right)^{22}$$

$$+ \cos 2\theta \Big\{ -(f_1+3f_2)\left(\frac{r}{a}\right)^2 + \frac{(95+15\nu)}{96}(f_1+15f_3)\left(\frac{r}{a}\right)^4 - \frac{(119+14\nu)}{4}f_2\left(\frac{r}{a}\right)^6 + \frac{(477+45\nu)}{16}f_2\left(\frac{r}{a}\right)^8$$

$$- \frac{(418+33\nu)}{28}f_2\left(\frac{r}{a}\right)^{10} + \frac{(1339+91\nu)}{448}f_2\left(\frac{r}{a}\right)^{12} - 7(1+\nu)p_2 - 12\nu h_2\left(\frac{r}{a}\right)^2 \Big\} \Big]$$

$$\frac{u}{R} = \left(\frac{a}{R}\right)^3 \Big[\frac{1}{2}(1-\nu)q_0\left(\frac{r}{a}\right) + \frac{(3-\nu)}{8}A\left(\frac{r}{a}\right)^3 + \frac{(5-\nu)}{24}B\left(\frac{r}{a}\right)^5 + \frac{(7-\nu)}{24}C\left(\frac{r}{a}\right)^7 + \frac{(9-\nu)}{40}D\left(\frac{r}{a}\right)^9$$

$$+ \frac{(11-\nu)}{8}E\left(\frac{r}{a}\right)^{11} + \frac{(13-\nu)}{56}F\left(\frac{r}{a}\right)^{13} + \frac{(15-\nu)}{56}G\left(\frac{r}{a}\right)^{15} + \frac{15(17-\nu)}{2}H\left(\frac{r}{a}\right)^{17} - \frac{9(19-\nu)}{4}I\left(\frac{r}{a}\right)^{19}$$

$$+ \frac{9(21-\nu)}{22}J\left(\frac{r}{a}\right)^{21} - \frac{3(23-\nu)}{88}K\left(\frac{r}{a}\right)^{23}$$

$$+ \cos 2\theta \Big\{ -\frac{1}{3}(f_1+3f_2)\left(\frac{r}{a}\right)^3 + \frac{(19+3\nu)}{96}(f_1+15f_2)\left(\frac{r}{a}\right)^5 - \frac{(17+2\nu)}{4}f_2\left(\frac{r}{a}\right)^7 + \frac{(53+5\nu)}{16}f_2\left(\frac{r}{a}\right)^9$$

$$- \frac{(38+3\nu)}{28}f_2\left(\frac{r}{a}\right)^{11} + \frac{(103+7\nu)}{448}f_2\left(\frac{r}{a}\right)^{13} - 7(1+\nu)p_2\left(\frac{r}{a}\right) - 4\nu h_2\left(\frac{r}{a}\right)^3 \Big\} \Big]$$

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The non-uniform part of $\hat{\sigma}_\theta - \nu \hat{\sigma}_r$ is

$$E\left(\frac{a}{R}\right)^2 \cos 2\theta \Big\{ -\frac{(15-\nu)}{96}(f_1+15f_2)\left(\frac{r}{a}\right)^4 + \frac{(14-\nu)}{4} f_2 \left(\frac{r}{a}\right)^6 - \frac{(45-3\nu)}{16} f_2 \left(\frac{r}{a}\right)^8 + \frac{(33-2\nu)}{28} f_2 \left(\frac{r}{a}\right)^{10} - \frac{(91-5\nu)}{488} f_2 \left(\frac{r}{a}\right)^{12}$$

$$+ 2(1+\nu)p_2 + 12 q_2 \left(\frac{r}{a}\right)^2 \Big\}\Big]$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \left(\frac{a}{R}\right)^2 \cos 2\theta \Big\{ \frac{1}{3}(f_1+3f_2)\left(\frac{r}{a}\right)^2 - \frac{(34+2\nu)}{96}(f_1+15f_2)\left(\frac{r}{a}\right)^4 + \frac{(31+\nu)}{4} f_2 \left(\frac{r}{a}\right)^6 - \frac{(98+2\nu)}{16} f_2 \left(\frac{r}{a}\right)^8$$

$$+ \frac{(71+\nu)}{28} f_2 \left(\frac{r}{a}\right)^{10} - \frac{(194+2\nu)}{448} f_2 \left(\frac{r}{a}\right)^{12} + 4(1+\nu)p_2 + 4(3+\nu)\left(\frac{r}{a}\right)^2 \Big\}.$$

$$\frac{v}{R} = \left(\frac{a}{R}\right)^3 \sin 2\theta \Big\{ \frac{1}{6}(f_1+3f_2)\left(\frac{r}{a}\right)^3 - \frac{(17+\nu)}{96}(f_1+15f_2)\left(\frac{r}{a}\right)^5 + \frac{(31+\nu)}{8} f_2 \left(\frac{r}{a}\right)^7 - \frac{(49+\nu)}{16} f_2 \left(\frac{r}{a}\right)^9$$

$$+ \frac{(71+\nu)}{56} f_2 \left(\frac{r}{a}\right)^{11} - \frac{(97+\nu)}{448} f_2 \left(\frac{r}{a}\right)^{13} + 2(1+\nu)p_2 + 2(3+\nu)q_2\left(\frac{r}{a}\right)^3 \Big\}$$

Now the non-uniform parts

$$\hat{rr}_a = \cos 2\theta\left\{-\frac{1}{96}f_1 - \frac{15}{448}f_2 - 2p_2\right\} \text{———} \left\{\frac{1}{2} - 6q_2 - 4s_2\right\}$$

$$\hat{\theta\theta}_a = \cos 2\theta\left\{-\frac{5}{32}f_1 - \frac{305}{448}f_2 + 2p_2 + 12r_2\right\} \text{———} \left\{6q_2 - \frac{1}{2}\right\}$$

$$\hat{r\theta}_a = \sin 2\theta\left\{-\frac{5}{96}f_1 - \frac{135}{448}f_2 + 2p_2 + 6r_2\right\} \text{———} \left\{-\frac{1}{2} - 6q_2 - 2s_2\right\}$$

$$\left(\frac{u}{R}\right)_a = \cos 2\theta\left\{-\frac{(13-3\nu)}{96}f_1 - \frac{43-85\nu}{448}f_2 - 2(1+\nu)p_2 - 4\nu r_2\right\} - \left\{\frac{1}{2}(1+\nu) + 2(1+\nu)q_2 + 4s_2\right\}$$

$$\left(\frac{v}{R}\right)_a = \sin 2\theta\left\{-\frac{(1+\nu)}{96}f_1 - \frac{131+35\nu}{448}f_2 + 2(1+\nu)p_2 + 2(3+\nu)r_2\right\} - \left\{2(1+\nu)q_2 - \frac{1}{2}(1+\nu)\right\}$$

$$\frac{.017.5}{1164.8} \qquad \frac{141.5}{1164.8}$$

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$$q_2 + 0.66666667 s_2 - 0.08333333 = \eta\{+0.33333333 p_2 + 0 + 0.00173611 f_1 + 0.00558036 f_2\}$$ **442**

$$q_2 + 0 - 0.08333333 = \eta\{+0.33333333 p_2 + 2 r_2 - 0.02604167 f_1 - 0.11346726 f_2\}$$

$$q_2 + 0.3333333 s_2 + 0.0833333 = \eta\{-0.3333333 p_2 - r_2 + 0.00868056 f_1 + 0.05022321 f_2\}$$

$$q_2 + 1.53846154 s_2 + 0.25000 = \eta\{-p_2 - 0.46153846 r_2 - 0.04847756 f_1 - 0.01502404 f_2\}$$

$$q_2 + 0 - 0.25000 = \eta\{+p_2 + 2.53846154 r_2 - 0.00520833 f_1 - 0.12148008 f_2\}$$

$$0.66666667 s_2 + 0 = \eta\{0 - 2 r_2 + 0.02777778 f_1 + 0.11904762 f_2\}$$

$$0.3333333 s_2 + 0.1666667 = \eta\{-0.6666667 p_2 - 3 r_2 + 0.03472222 f_1 + 0.16369047 f_2\}$$

$$1.20512821 s_2 + 0.1666667 = \eta\{-0.6666667 p_2 + 0.53846154 r_2 - 0.05715812 f_1 - 0.06524725 f_2\}$$

$$1.53846154 s_2 + 0.50000 = \eta\{-2 p_2 - 3 r_2 - 0.04326923 f_1 + 0.10645604 f_2\}$$

$$s_2 + 0 = \eta\{0 - 3 r_2 + 0.04166667 f_1 + 0.17857143 f_2\}$$

$$s_2 + 0.50000 = \eta\{-2 p_2 - 9 r_2 + 0.10416667 f_1 + 0.49107141 f_2\}$$

$$s_2 + 0.13829787 = \eta\{-0.55319149 p_2 + 0.44680851 r_2 - 0.04742908 f_1 - 0.05414133 f_2\}$$

$$s_2 + 0.325000 = \eta\{-1.3 p_2 - 1.95 r_2 - 0.028125000 f_1 + 0.06919643 f_2\}$$

$$0.50000000 = \eta\{-2 p_2 - 6 r_2 + 0.06250000 f_1 + 0.31250000 f_2\}$$

$$0.36170213 = \eta\{-1.44680851 p_2 - 9.44680851 r_2 + 0.15159575 f_1 + 0.54521274 f_2\}$$

$$0.18670213 = \eta\{-0.74680851 p_2 - 2.39680851 r_2 + 0.01930408 f_1 + 0.12333771 f_2\}$$

$$0.25000000 = \eta \left\{ - p_2 - 3 r_2 + 0.03125000 f_1 + 0.15625000 f_2 \right\}$$

$$0.25000000 = \eta \left\{ - p_2 - 6.52941176 \, r_2 + 0.10477942 f_1 + 0.37683822 f_2 \right\}$$

$$0.2500000 = \eta \left\{ - p_2 - 3.20940171 \, r_2 + 0.02584877 f_1 + 0.16515313 f_2 \right\}$$

$$3.52941176 \, r_2 = 0.07352942 f_1 + 0.22058822 f_2$$

$$3.32001005 \, r_2 = 0.07893065 f_1 + 0.21168509 f_2$$

$$3 r_2 = 0.06250000 f_1 + 0.1875000 f_2$$

$$3 r_2 = 0.07132266 f_1 + 0.19128113 f_2$$

$$0 = 0.00882266 f_1 + 0.00378113 f_2$$

New Calculation

For the circular region,

$$\frac{w_0}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left\{1-\left(\frac{r}{a}\right)^2\sin^2\theta\right\}$$

$$\frac{w}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left\{1-\left(\frac{r}{a}\right)^2\sin^2\theta - f_1\left[1-\left(\frac{r}{a}\right)^2\right]^2 - f_2\left[1-\left(\frac{r}{a}\right)^2\right]^4\right\}$$

$$\frac{1}{r}\frac{\partial w}{\partial r} = \frac{1}{R}\left\{-\sin^2\theta + 2f_1\left[1-\left(\frac{r}{a}\right)^2\right] + 4f_2\left[1-\left(\frac{r}{a}\right)^2\right]^3\right\}$$

$$\frac{1}{r}\frac{\partial w_0}{\partial r} = \frac{1}{R}\left\{-\sin^2\theta\right\}$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{1}{R}\left\{-\sin^2\theta + 2f_1\left[1-3\left(\frac{r}{a}\right)^2\right] + 4f_2\left[1-\left(\frac{r}{a}\right)^2\right]^2\left[1-7\left(\frac{r}{a}\right)^2\right]\right\}$$

$$\frac{\partial^2 w_0}{\partial r^2} = \frac{1}{R}\left\{-\sin^2\theta\right\}.$$

$$\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} = \frac{1}{R}\left\{-\cos 2\theta\right\}$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\}$$

$$=\frac{1}{R^2}\Bigg[(1-\cos 2\theta)\left\{2f_1\left[1-2\left(\frac{r}{a}\right)^2\right]+4f_2\left[1-\left(\frac{r}{a}\right)^2\right]^2\left[1-4\left(\frac{r}{a}\right)^2\right]\right\}$$

$$-\left\{4f_1^2\left[1-\left(\frac{r}{a}\right)^2\right]\left[1-3\left(\frac{r}{a}\right)^2\right]+8f_1f_2\left[1-\left(\frac{r}{a}\right)^2\right]^3\left[2-10\left(\frac{r}{a}\right)^2\right]+16f_2^2\left[1-\left(\frac{r}{a}\right)^2\right]^5\left[1-7\left(\frac{r}{a}\right)^2\right]\right\}\Bigg]$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial \theta^2}-\frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right\}=\frac{1}{R^2}\cos 2\theta\left\{2f_1\left[1-3\left(\frac{r}{a}\right)^2\right]+4f_2\left[1-\left(\frac{r}{a}\right)^2\right]^2\left[1-7\left(\frac{r}{a}\right)^2\right]\right\}$$

The terms for the particular integral is then, <u>multiplied by R^2</u>

$$2f_1\left[1-2\left(\frac{r}{a}\right)^2\right]+4f_2\left[1-\left(\frac{r}{a}\right)^2\right]^2\left[1-4\left(\frac{r}{a}\right)^2\right]-4f_1^2\left[1-\left(\frac{r}{a}\right)^2\right]\left[1-3\left(\frac{r}{a}\right)^2\right]-16f_1f_2\left[1-\left(\frac{r}{a}\right)^2\right]^3\left[1-5\left(\frac{r}{a}\right)^2\right]$$

$$-16f_2^2\left[1-\left(\frac{r}{a}\right)^2\right]^5\left[1-7\left(\frac{r}{a}\right)^2\right]$$

$$+\cos 2\theta\left\{-2f_1\left(\frac{r}{a}\right)^2-12f_2\left[1-\left(\frac{r}{a}\right)^2\right]^2\left(\frac{r}{a}\right)^2\right\}$$

1	$\left(\frac{r}{a}\right)^2$	$\left(\frac{r}{a}\right)^4$	$\left(\frac{r}{a}\right)^6$	$\left(\frac{r}{a}\right)^8$	$\left(\frac{r}{a}\right)^{10}$	$\left(\frac{r}{a}\right)^{12}$
$+2f_1$	$-4f_1$					
$+4f_2$	$-8f_2$ $-16f_2$	$+4f_2$ $+32f_2$	$-16f_2$			
$-4f_1^2$	$+16f_1^2$	$-12f_1^2$				
$-16f_1f_2$	$+48f_1f_2$ $+80f_1f_2$	$-48f_1f_2$ $-240f_1f_2$	$+16f_1f_2$ $+240f_1f_2$	$+80f_1f_2$		
$-16f_2^2$	$+80f_2^2$ $+112f_2^2$	$-160f_2^2$ $-560f_2^2$	$+160f_2^2$ $+1120f_2^2$	$-80f_2^2$ $-1120f_2^2$	$+16f_2^2$ $+560f_2^2$	$-112f_2^2$
$\frac{1}{16\cdot 4}$	$\frac{1}{36\cdot 16}$	$\frac{1}{64\cdot 36}$	$\frac{1}{100\cdot 64}$	$\frac{1}{144\cdot 100}$	$\frac{1}{196\cdot 144}$	$\frac{1}{256\cdot 196}$

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$\left(\frac{r}{a}\right)^2$	$\left(\frac{r}{a}\right)^4$	$\left(\frac{r}{a}\right)^6$
$-2f_1$ $-12f_2$	$+24f_2$	$-12f_2$
$\frac{1}{32\cdot 12}$	$\frac{1}{60\cdot 32}$	$\frac{1}{96\cdot 60}$

$$\frac{\Phi}{R^2} = E\left(\frac{a}{R}\right)^4 \Big[\frac{1}{4} q_0 \left(\frac{r}{a}\right)^2 + \frac{1}{32}\left(f_1 + 2f_2 - 2f_1^2 - 8f_1 f_2 - 8f_2^2\right)\left(\frac{r}{a}\right)^4 + \frac{1}{144}\left(-f_1 - 6f_2 + 4f_1^2 + 32 f_1 f_2 + 48 f_2^2\right)\left(\frac{r}{a}\right)^6$$

$$+ \frac{1}{192}\left(3f_2 - f_1^2 - 24 f_1 f_2 - 60 f_2^2\right)\left(\frac{r}{a}\right)^8 + \frac{1}{400}\left(-f_2 + 16 f_1 f_2 + 80 f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$+ \frac{1}{180}\left(f_1 f_2 - 15 f_2^2\right)\left(\frac{r}{a}\right)^{12} + \frac{1}{49} f_2^2 \left(\frac{r}{a}\right)^{14} - \frac{1}{448} f_2^2 \left(\frac{r}{a}\right)^{16}$$

$$+ \cos 2\theta \left\{ -\frac{1}{192}\left(f_1 + 6f_2\right)\left(\frac{r}{a}\right)^6 + \frac{1}{80} f_2 \left(\frac{r}{a}\right)^8 - \frac{1}{480} f_2 \left(\frac{r}{a}\right)^{10} + p_2 \left(\frac{r}{a}\right)^2 + r_2 \left(\frac{r}{a}\right)^4 \right\} \Big]$$

$$\frac{1}{r}\frac{\partial \Phi}{\partial r} = E\left(\frac{a}{R}\right)^2 \Big[\frac{1}{2} q_0 + \frac{1}{8}\left(f_1 + 2f_2 - 2f_1^2 - 8 f_1 f_2 - 8 f_2^2\right)\left(\frac{r}{a}\right)^2 + \frac{1}{24}\left(-f_1 - 6f_2 + 4f_1^2 + 32 f_1 f_2 + 48 f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$+ \frac{1}{24}\left(3f_2 - f_1^2 - 24 f_1 f_2 - 60 f_2^2\right)\left(\frac{r}{a}\right)^6 + \frac{1}{40}\left(-f_2 + 16 f_1 f_2 + 80 f_2^2\right)\left(\frac{r}{a}\right)^8 + \frac{1}{15}\left(f_1 f_2 - 15 f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$+ \frac{2}{7} f_2^2 \left(\frac{r}{a}\right)^{12} - \frac{1}{28} f_2^2 \left(\frac{r}{a}\right)^{14} + \cos 2\theta \left\{ -\frac{6}{192}\left(f_1 + 6f_2\right)\left(\frac{r}{a}\right)^4 + \frac{8}{80} f_2 \left(\frac{r}{a}\right)^6 - \frac{10}{480} f_2 \left(\frac{r}{a}\right)^8 \right.$$

$$\left. + 2p_2 + 4r_2 \left(\frac{r}{a}\right)^2 \right\} \Big]$$

$$\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = E\left(\frac{a}{R}\right)^2 \cos 2\theta \left\{ + \frac{4}{192}\left(f_1 + 6f_2\right)\left(\frac{r}{a}\right)^4 - \frac{4}{80} f_2 \left(\frac{r}{a}\right)^6 + \frac{4}{480} f_2 \left(\frac{r}{a}\right)^{10} - 4p_2 - 4r_2 \left(\frac{r}{a}\right)^4 \right\}$$

$$\widehat{rr} = E\left(\frac{a}{R}\right)^2\Big[\frac{1}{2}q_0 + \frac{1}{8}(f_1 + 2f_2 - 2f_1^2 - 8f_1f_2 - 8f_2^2)\left(\frac{r}{a}\right)^2 + \frac{1}{24}(-f_1 - 6f_2 + 4f_1^2 + 32f_1f_2 + 48f_2^2)\left(\frac{r}{a}\right)^4$$

$$+ \frac{1}{24}(3f_2 - f_1^2 - 24f_1f_2 - 60f_2^2)\left(\frac{r}{a}\right)^6 + \frac{1}{40}(-f_2 + 16f_1f_2 + 80f_2^2)\left(\frac{r}{a}\right)^8 + \frac{1}{15}(f_1f_2 - 15f_2^2)\left(\frac{r}{a}\right)^{10}$$

$$+ \frac{2}{7}f_2^2\left(\frac{r}{a}\right)^{12} - \frac{1}{28}f_2^2\left(\frac{r}{a}\right)^{14} + \cos 2\theta\Big\{-\frac{1}{96}(f_1 + 6f_2)\left(\frac{r}{a}\right)^4 + \frac{1}{20}f_2\left(\frac{r}{a}\right)^6 - \frac{1}{80}f_2\left(\frac{r}{a}\right)^{10} - 2p_2\Big\}$$

$$\widehat{\theta\theta} = E\left(\frac{a}{R}\right)^2\Big[\frac{1}{2}q_0 + \frac{3}{8}(f_1 + 2f_2 - 2f_1^2 - 8f_1f_2 - 8f_2^2)\left(\frac{r}{a}\right)^2 + \frac{5}{24}(-f_1 - 6f_2 + 4f_1^2 + 32f_1f_2 + 48f_2^2)\left(\frac{r}{a}\right)^4$$

$$+ \frac{7}{24}(3f_2 - f_1^2 - 24f_1f_2 - 60f_2^2)\left(\frac{r}{a}\right)^6 + \frac{9}{40}(-f_2 + 16f_1f_2 + 80f_2^2)\left(\frac{r}{a}\right)^8 + \frac{11}{15}(f_1f_2 - 15f_2^2)\left(\frac{r}{a}\right)^{10}$$

$$+ \frac{26}{7}f_2^2\left(\frac{r}{a}\right)^{12} - \frac{15}{28}f_2^2\left(\frac{r}{a}\right)^{14} + \cos 2\theta\Big\{-\frac{15}{96}(f_1 + 6f_2)\left(\frac{r}{a}\right)^4 + \frac{14}{20}f_2\left(\frac{r}{a}\right)^6 - \frac{15}{80}f_2\left(\frac{r}{a}\right)^8$$

$$+ 2p_2 + 12r_2\left(\frac{r}{a}\right)^2\Big\}$$

$$\widehat{r\theta} = E\left(\frac{a}{R}\right)^2\sin 2\theta\Big\{-\frac{5}{96}(f_1 + 6f_2)\left(\frac{r}{a}\right)^4 + \frac{7}{40}f_2\left(\frac{r}{a}\right)^6 - \frac{3}{80}f_2\left(\frac{r}{a}\right)^8 + 2p_2 + 6r_2\left(\frac{r}{a}\right)^2\Big\}$$

$$\widehat{rr}-\widehat{\theta\theta}=E\left(\frac{a}{R}\right)^2\Big[\frac{1}{2}(1-\nu)g_0+\frac{(1-3\nu)}{8}\left(f_1+2f_2-2f_1^2-8f_1f_2-8f_2^2\right)\left(\frac{r}{a}\right)^2+\frac{(1-5\nu)}{24}\left(-f_1-6f_2+4f_1^2+32f_1f_2+48f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$+\frac{(1-7\nu)}{24}\left(3f_2-f_1^2-24f_1f_2-60f_2^2\right)\left(\frac{r}{a}\right)^6+\frac{(1-9\nu)}{40}\left(-f_2+16f_1f_2+80f_2^2\right)\left(\frac{r}{a}\right)^8+\frac{(1-11\nu)}{15}\left(f_1f_2-15f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$+\frac{2(1-13\nu)}{7}f_2^2\left(\frac{r}{a}\right)^{12}-\frac{(1-15\nu)}{28}f_2^2\left(\frac{r}{a}\right)^{14}+\cos 2\theta\Big\{-\frac{(1-15\nu)}{96}\left(f_1+6f_2\right)\left(\frac{r}{a}\right)^4+\frac{(1-14\nu)}{20}f_2\left(\frac{r}{a}\right)^6-\frac{(1-15\nu)}{80}f_2\left(\frac{r}{a}\right)^8$$

$$-2(1+\nu)p_2-12\nu r_2\left(\frac{e}{a}\right)^2\Big\}$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2-\frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2=\frac{1}{2}\left(\frac{r}{R}\right)^2\Big[(\cos 2\theta-1)\Big\{2f_1\left[1-\left(\frac{r}{a}\right)^2\right]+4f_2\left[1-\left(\frac{r}{a}\right)^2\right]^3\Big\}$$

$$+4f_1^2\left[1-\left(\frac{r}{a}\right)^2\right]^2+16f_2f_1\left[1-\left(\frac{r}{a}\right)^2\right]^4+16f_2^2\left[1-\left(\frac{r}{a}\right)^2\right]^6\Big]$$

$$=\left(\frac{a}{R}\right)^2\Big[-f_1\left[1-\left(\frac{r}{a}\right)^2\right]\left(\frac{r}{a}\right)^2-2f_2\left[1-\left(\frac{r}{a}\right)^2\right]^3\left(\frac{r}{a}\right)^2+2f_1^2\left[1-\left(\frac{r}{a}\right)^2\right]^2\left(\frac{r}{a}\right)^2+8f_2f_1\left[1-\left(\frac{r}{a}\right)^2\right]^4\left(\frac{r}{a}\right)^2+8f_2^2\left[1-\left(\frac{r}{a}\right)^2\right]^6\left(\frac{r}{a}\right)^2$$

$$+\cos 2\theta\Big\{f_1\left[1-\left(\frac{r}{a}\right)^2\right]\left(\frac{r}{a}\right)^2+2f_2\left[1-\left(\frac{r}{a}\right)^2\right]^3\left(\frac{r}{a}\right)^2\Big\}\Big]$$

$$=\left(\frac{a}{R}\right)^2\Big[-\left(f_1+2f_2-2f_1^2-8f_1f_2-8f_2^2\right)\left(\frac{r}{a}\right)^2-\left(-f_1-6f_2+4f_1^2+32f_1f_2+48f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$-2\left(3f_2-f_1^2-24f_1f_2-60f_2^2\right)\left(\frac{r}{a}\right)^6-9\left(-f_2+16f_1f_2+80f_2^2\right)\left(\frac{r}{a}\right)^8-8\left(f_1f_2-15f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$-48f_2^2\left(\frac{r}{a}\right)^{12}+8f_2^2\left(\frac{r}{a}\right)^{14}+\cos 2\theta\Big\{\left(f_1+2f_2\right)\left(\frac{r}{a}\right)^2-\left(f_1+6f_2\right)\left(\frac{r}{a}\right)^4+6f_2\left(\frac{r}{a}\right)^6-2f_2\left(\frac{r}{a}\right)^8\Big\}\Big]$$

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$$\frac{\partial u}{\partial r} = \left(\frac{a}{R}\right)^2 \Big[\frac{1}{2}(1-\nu)q_0 + \frac{3(3-\nu)}{8}(f_1+2f_2-2f_1^2-8f_1f_2-8f_2^2)\left(\frac{r}{a}\right)^2 + \frac{5(5-\nu)}{24}(-f_1-6f_2+4f_1^2+32f_1f_2+48f_2^2)\left(\frac{r}{a}\right)^4$$

$$+ \frac{7(7-\nu)}{24}(3f_2-f_2^2-24f_1f_2-60f_2^2)\left(\frac{r}{a}\right)^6 + \frac{9(9-\nu)}{40}(-f_2+16f_1f_2+80f_2^2)\left(\frac{r}{a}\right)^8 + \frac{11(11-\nu)}{15}(f_1f_2-15f_2^2)\left(\frac{r}{a}\right)^{10}$$

$$+ \frac{26(13-\nu)}{7}f_2^2\left(\frac{r}{a}\right)^{12} - \frac{15(15-\nu)}{28}f_2^2\left(\frac{r}{a}\right)^{14} + \cos 2\theta \Big\{ -(f_1+2f_2)\left(\frac{r}{a}\right)^2 + \frac{(95+15\nu)}{96}(f_1+6f_2)\left(\frac{r}{a}\right)^4 - \frac{(119+14\nu)}{20}f_2\left(\frac{r}{a}\right)^6$$

$$+ \frac{(159+15\nu)}{80}f_2\left(\frac{r}{a}\right)^8$$

$$- 2(1+\nu)p_2 - 12\nu r_2\left(\frac{r}{a}\right)^2 \Big\} \Big]$$

$$\frac{u}{r} = \left(\frac{a}{R}\right)^3 \Big[\frac{1}{2}(1-\nu)q_0\left(\frac{r}{a}\right) + \frac{(3-\nu)}{8}(f_1+2f_2-2f_1^2-8f_1f_2-8f_2^2)\left(\frac{r}{a}\right)^3 + \frac{(5-\nu)}{24}(-f_1-6f_2+4f_1^2+32f_1f_2+48f_2^2)\left(\frac{r}{a}\right)^5$$

$$+ \frac{(7-\nu)}{24}(3f_2-f_2^2-24f_1f_2-60f_2^2)\left(\frac{r}{a}\right)^7 + \frac{(9-\nu)}{40}(-f_2+16f_1f_2+80f_2^2)\left(\frac{r}{a}\right)^9 + \frac{(11-\nu)}{15}(f_1f_2-15f_2^2)\left(\frac{r}{a}\right)^{11}$$

$$+ \frac{2(13-\nu)}{7}f_2^2\left(\frac{r}{a}\right)^{12} - \frac{(15-\nu)}{28}f_2^2\left(\frac{r}{a}\right)^{14} + \cos 2\theta \Big\{ -\frac{1}{3}(f_1+2f_2)\left(\frac{r}{a}\right)^3 + \frac{(19+3\nu)}{96}(f_1+6f_2)\left(\frac{r}{a}\right)^5 - \frac{(17+2\nu)}{20}f_2\left(\frac{r}{a}\right)^7$$

$$+ \frac{(53+5\nu)}{240}f_2\left(\frac{r}{a}\right)^9 - 2(1+\nu)p_2\left(\frac{r}{a}\right) - 4\nu r_2\left(\frac{r}{a}\right)^3 \Big\} \Big]$$

Non-uniform part of $\widehat{\theta\theta} - \nu\widehat{rr}$ is

$$E\left(\frac{a}{R}\right)^2 \cos 2\theta \left\{ -\frac{(15-\nu)}{96}(f_1+6f_2)\left(\frac{r}{a}\right)^4 + \frac{(14-\nu)}{20} f_2 \left(\frac{r}{a}\right)^6 - \frac{(45-3\nu)}{240} f_2 \left(\frac{r}{a}\right)^8 + 2(1+\nu)p_2 + 12 r_2 \left(\frac{r}{a}\right)^2 \right\}$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \left(\frac{a}{R}\right)^2 \cos 2\theta \left\{ \frac{1}{3}(f_1+2f_2)\left(\frac{r}{a}\right)^2 - \frac{(34+2\nu)}{96}(f_1+6f_2)\left(\frac{r}{a}\right)^4 + \frac{(31+\nu)}{20} f_2 \left(\frac{r}{a}\right)^6 - \frac{(98+2\nu)}{240} f_2 \left(\frac{r}{a}\right)^9 \right.$$

$$\left. + 4(1+\nu)p_2 + 4(3+\nu) r_2 \left(\frac{r}{a}\right)^2 \right\} \Big]$$

$$\frac{v}{R} = \left(\frac{a}{R}\right)^3 \sin 2\theta \left\{ \frac{1}{6}(f_1+2f_2)\left(\frac{r}{a}\right)^3 - \frac{(17+\nu)}{96}(f_1+6f_2)\left(\frac{r}{a}\right)^5 + \frac{(31+\nu)}{40} f_2 \left(\frac{r}{a}\right)^7 - \frac{(49+\nu)}{240} f_2 \left(\frac{r}{a}\right)^9 \right.$$

$$\left. + 2(1+\nu)p_2\left(\frac{r}{a}\right) + 2(3+\nu) r_2 \left(\frac{r}{a}\right)^3 \right\} \Big]$$

$$\widehat{rr}_a = \cos 2\theta \left\{ -\frac{1}{96} f_1 - \frac{1}{40} f_2 - 2p_2 \right\}$$

$$\widehat{\theta\theta}_a = \cos 2\theta \left\{ -\frac{5}{32} f_1 - \frac{17}{40} f_2 + 2p_2 + 12 r_2 \right\}$$

$$\widehat{r\theta}_a = \sin 2\theta \left\{ -\frac{5}{96} f_1 - \frac{7}{40} f_2 + 2p_2 + 6 r_2 \right\}$$

$$\left(\frac{u}{R}\right)_a = \cos 2\theta \left\{ -\frac{(13-3\nu)}{96} f_1 - \frac{26-26\nu}{140} f_2 - 2(1+\nu)p_2 - 12\nu r_2 \right\}$$

$$\left(\frac{v}{R}\right)_a = \sin 2\theta \left\{ -\frac{(1+\nu)}{96} f_1 - \frac{38+10\nu}{140} f_2 + 2(1+\nu)p_2 + 2(3+\nu) r_2 \right\}$$

$$\frac{1}{2} - 6q_2 - 4s_2 = \eta\left\{ -\frac{1}{96} f_1 - \frac{1}{40} f_2 - 2p_2 \right\}$$

$$6q_2 - \frac{1}{2} = \eta\left\{ -\frac{5}{32} f_1 - \frac{17}{40} f_2 + 2p_2 + 12r_2 \right\}$$

$$\frac{13}{2496} \qquad \frac{41}{6240}$$

$$\frac{1}{2} + 6q_2 + 2s_2 = \eta\left\{ +\frac{5}{96} f_1 + \frac{7}{40} f_2 - 2p_2 - 6r_2 \right\}$$

$$\frac{1}{2}(1+\nu) + 2(1+\nu)q_2 + 4s_2 = \eta\left\{ -\frac{(13-3\nu)}{96} f_1 - \frac{26(1-\nu)}{240} f_2 - 2(1+\nu)p_2 - 4\nu r_2 \right\}$$

$$2(1+\nu)q_2 - \frac{1}{2}(1+\nu) = \eta\left\{ -\frac{(1+\nu)}{96} f_1 - \frac{38+10\nu}{240} f_2 + 2(1+\nu)p_2 + 2(3+\nu)r_2 \right\}$$

$$q_2 + 0.66666667 s_2 - 0.08333333 = \eta\left\{ +0.3333333 p_2 + 0 + 0.0017361 f_1 + 0.00416667 f_2 \right\}$$

$$q_2 \quad 0 \quad - 0.08333333 = \eta\left\{ +0.3333333 p_2 + 2r_2 - 0.02604167 f_1 - 0.07083333 f_2 \right\}$$

$$q_2 + 0.3333333 s_2 + 0.08333333 = \eta\left\{ -0.3333333 p_2 - r_2 + 0.00868056 f_1 + 0.02916667 f_2 \right\}$$

$$q_2 + 1.53846154 s_2 + 0.2500000 = \eta\left\{ -p_2 - 0.46153846 r_2 - 0.0484775 f_1 - 0.02916667 f_2 \right\}$$

$$q_2 + \quad 0 \quad - 0.2500000 = \eta\left\{ +p_2 + 2.53846154 r_2 - 0.00520833 f_1 - 0.06570513 f_2 \right\}$$

$$0.66666667 s_2 + \quad 0 \quad = \eta\left\{ 0 \quad -2r_2 + 0.02777778 f_1 + 0.07500000 f_2 \right\}$$

$$0.333333 s_2 + 0.1666667 = \eta\left\{ -0.666667 p_2 - 3r_2 + 0.03472222 f_1 + 0.10000000 f_2 \right\}$$

$$1.20512821 s_2 + 0.166667 = \eta\left\{ -0.666667 p_2 + 0.53846154 r_2 - 0.05715812 f_1 - 0.058333333 f_2 \right\}$$

$$1.53846154 s_2 + 0.50000 = \eta\left\{ -2p_2 - 3r_2 - 0.04326923 f_1 + 0.03653846 f_2 \right\}$$

$$s_2 + \quad 0 = \eta\{\ 0 \quad - 3r_2 + 0.04166667 f_1 + 0.11250000 f_2\}$$
$$s_2 + 0.50000 = \eta\{-2p_2 \quad - 9r_2 + 0.10416667 f_1 + 0.30000000 f_2\}$$
$$s_2 + 0.13829787 = \eta\{-0.55319149 p_2 + 0.44680851 r_2 - 0.04742908 f_1 - 0.04840425 f_2\}$$
$$s_2 + 0.3250000 = \eta\{-1.3 p_2 - 1.95 r_2 - 0.02812500 f_1 + 0.02375000 f_2\}$$

$$0.5000000 = \eta\{-2p_2 - 6r_2 + 0.06250000 f_1 + 0.18750000 f_2\}$$
$$0.36170213 = \eta\{-1.44680851 p_2 - 9.44680851 r_2 + 0.15159575 f_1 + 0.34840425 f_2\}$$
$$0.18670213 = \eta\{-0.74680851 p_2 - 2.39680851 r_2 + 0.01930408 f_1 + 0.07215425 f_2\}$$

$$0.2500000 = \eta\{-p_2 - 3r_2 + 0.03125000 f_1 + 0.09375000 f_2\}$$
$$0.2500000 = \eta\{-p_2 - 6.52941176 r_2 + 0.10477942 f_1 + 0.24080882 f_2\}$$
$$0.250000 = \eta\{-p_2 - 3.20940171 r_2 + 0.02584877 f_1 + 0.09661680 f_2\}$$

$$3.52941176\, r_2 = 0.07352942 f_1 + 0.14705882 f_2$$
$$3.32001005\, r_2 = 0.07893065 f_1 + 0.14419202 f_2$$

$$3r_2 = 0.06250000 f_1 + 0.12500000 f_2$$
$$3r_2 = 0.07132266 f_1 + 0.13029360 f_2$$

$$0 = 0.00882266 f_1 + 0.00529360 f_2$$

$$f_2 = -\frac{5}{3} f_1$$

failure !!!

a)

$$\left(\frac{w}{R}\right)_0 = \frac{1}{2}\left(\frac{a}{R}\right)^2\left\{1-\left(\frac{r}{a}\right)^2\sin^2\theta\right\}$$

$$\left(\frac{w}{R}\right) = \frac{1}{2}\left(\frac{a}{R}\right)^2\left\{1-\left(\frac{r}{a}\right)^2\sin^2\theta - f_4\left[1-\left(\frac{r}{a}\right)^2\right]^6\right\}$$

$$\frac{1}{r}\frac{\partial w}{\partial r} = \frac{1}{2}\frac{1}{R}\left\{-2\sin^2\theta + 12 f_4\left[1-\left(\frac{r}{a}\right)^2\right]^5\right\}$$

$$\frac{1}{r}\frac{\partial w_0}{\partial r} = \frac{1}{2}\frac{1}{R}\left\{-2\sin^2\theta\right\}$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{1}{2}\frac{1}{R}\left\{-2\sin^2\theta + 12 f_4\left[1-\left(\frac{r}{a}\right)^2\right]^5 - 120 f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4\left(\frac{r}{a}\right)^2\right\}$$

$$= \frac{1}{2}\frac{1}{R}\left\{-2\sin^2\theta + 12 f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4\left[1-\left(\frac{r}{a}\right)^2 - 10\left(\frac{r}{a}\right)^2\right]\right\}$$

$$= \frac{1}{2}\frac{1}{R}\left\{-2\sin^2\theta + 12 f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4\left[1-11\left(\frac{r}{a}\right)^2\right]\right\}$$

$$\frac{\partial^2 w_0}{\partial r^2} = \frac{1}{2}\frac{1}{R}\left\{-2\sin^2\theta\right\}$$

$$\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} = \frac{1}{R}\left(-\cos 2\theta\right)$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\}$$ 6)

$$=\frac{1}{R^2}(\sin^2\theta)^2-\frac{1}{R^2}\left\{\sin^2\theta-6f_4\left[1-\left(\frac{r}{a}\right)^2\right]^5\right\}\left\{\sin^2\theta-6f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4\left[1-11\left(\frac{r}{a}\right)^2\right]\right\}$$

$$=\frac{1}{R^2}\left\{\underline{\underline{(1-\cos 2\theta)\cdot 6f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4\left[1-6\left(\frac{r}{a}\right)^2\right]}}-36f_4^2\left[1-\left(\frac{r}{a}\right)^2\right]^9\left[1-11\left(\frac{r}{a}\right)^2\right]\right\}$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial \theta^2}-\frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right\}$$

$$=\frac{1}{R^2}\cos 2\theta\left\{\underline{\underline{6f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4\left[1-11\left(\frac{r}{a}\right)^2\right]}}\right\}$$

The non-uniform terms in particular integral must be derived from

$$\cos 2\theta\left[-6f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4 5\left(\frac{r}{a}\right)^2\right]=-\left[30f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4\left(\frac{r}{a}\right)^2\right]\cos 2\theta$$

$$=-30f_4\cos 2\theta\left\{\left(\frac{r}{a}\right)^2-4\left(\frac{r}{a}\right)^4+6\left(\frac{r}{a}\right)^6-4\left(\frac{r}{a}\right)^8+\left(\frac{r}{a}\right)^{10}\right\}$$

$$\frac{\Phi}{R^2}=-30f_4\cos 2\theta\left\{\frac{1}{32\cdot 12}\left(\frac{r}{a}\right)^6-\frac{4}{60\cdot 32}\left(\frac{r}{a}\right)^8+\frac{6}{96\cdot 60}\left(\frac{r}{a}\right)^{10}-\frac{4}{140\cdot 96}\left(\frac{r}{a}\right)^{12}+\frac{1}{192\cdot 140}\left(\frac{r}{a}\right)^{14}\right\}E\left(\frac{a}{R}\right)^7$$

$$\frac{1}{r}\frac{\partial\Phi}{\partial r} = -30 f_4 \cos 2\theta \left\{ \frac{1}{64}\left(\frac{r}{a}\right)^4 - \frac{1}{60}\left(\frac{r}{a}\right)^6 + \frac{1}{96}\left(\frac{r}{a}\right)^8 - \frac{1}{280}\left(\frac{r}{a}\right)^{10} + \frac{1}{1920}\left(\frac{r}{a}\right)^{12} \right\} E\left(\frac{a}{R}\right)^2 \qquad \text{c)}$$

$$\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = +30 f_4 \cos 2\theta \left\{ \frac{1}{96}\left(\frac{r}{a}\right)^4 - \frac{1}{120}\left(\frac{r}{a}\right)^6 + \frac{1}{240}\left(\frac{r}{a}\right)^8 - \frac{1}{840}\left(\frac{r}{a}\right)^{10} + \frac{1}{48\times 140}\left(\frac{r}{a}\right)^{12} \right\} E\left(\frac{a}{R}\right)^2$$

$$\widehat{rr} = -30 f_4 \cos 2\theta \left\{ \frac{1}{192}\left(\frac{r}{a}\right)^4 - \frac{1}{120}\left(\frac{r}{a}\right)^6 + \frac{1}{160}\left(\frac{r}{a}\right)^8 - \frac{1}{420}\left(\frac{r}{a}\right)^{10} + \frac{1}{192\cdot 14}\left(\frac{r}{a}\right)^{12} \right\} E\left(\frac{a}{R}\right)^2$$

$$\widehat{\theta\theta} = -30 f_4 \cos 2\theta \left\{ \frac{5}{64}\left(\frac{r}{a}\right)^4 - \frac{7}{60}\left(\frac{r}{a}\right)^6 + \frac{9}{96}\left(\frac{r}{a}\right)^8 - \frac{11}{280}\left(\frac{r}{a}\right)^{10} + \frac{13}{1920}\left(\frac{r}{a}\right)^{12} \right\} E\left(\frac{a}{R}\right)^2$$

$$\widehat{r\theta} = -30 f_4 \sin 2\theta \left\{ \frac{5}{32\cdot 6}\left(\frac{r}{a}\right)^4 - \frac{\overset{(\times)}{7}}{60\cdot 8}\left(\frac{r}{a}\right)^6 + \frac{9}{96\times 5}\left(\frac{r}{a}\right)^8 - \frac{11}{140\cdot 12}\left(\frac{r}{a}\right)^{10} + \frac{13}{192\times 70}\left(\frac{r}{a}\right)^{12} \right\} E\left(\frac{a}{R}\right)^2$$

$$\widehat{rr} - \nu\widehat{\theta\theta} = -30 f_4 \cos 2\theta \left\{ \frac{(1-15\nu)}{192}\left(\frac{r}{a}\right)^4 - \frac{(1-14\nu)}{120}\left(\frac{r}{a}\right)^6 + \frac{(1-15\nu)}{160}\left(\frac{r}{a}\right)^8 - \frac{(2-33\nu)}{840}\left(\frac{r}{a}\right)^{10} + \frac{(5-91\nu)}{13440}\left(\frac{r}{a}\right)^{12} \right\} E\left(\frac{a}{R}\right)^2$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2 = \frac{1}{2}\left(\frac{a}{R}\right)^2 \left\{ -(1-\cos 2\theta)\, 6 f_4 \left[1-\left(\frac{r}{a}\right)^2\right]^5 + 36 f_4^2 \left[1-\left(\frac{r}{a}\right)^2\right]^{10} \right\}$$

$$= \frac{1}{2}\left(\frac{a}{R}\right)^2 \left\{ \cos 2\theta \cdot 6 f_4 \left[1-\left(\frac{r}{a}\right)^2\right]^5 \left(\frac{r}{a}\right)^2 + \cdots\cdots \right\}$$

The non-uniform part of $\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2$ d)

$$3f_4\cos 3\theta\left\{\left(\frac{r}{a}\right)^2 - 5\left(\frac{r}{a}\right)^4 + 10\left(\frac{r}{a}\right)^6 - 10\left(\frac{r}{a}\right)^8 + 5\left(\frac{r}{a}\right)^{10} - \left(\frac{r}{a}\right)^{12}\right\}\left(\frac{a}{R}\right)^2$$

$$\frac{\partial u}{\partial r} = -30f_4\cos 2\theta\left\{\frac{1}{10}\left(\frac{r}{a}\right)^2 - \frac{(95+15\nu)}{192}\left(\frac{r}{a}\right)^4 + \frac{(119+14\nu)}{120}\left(\frac{r}{a}\right)^6 - \frac{(159+15\nu)}{160}\left(\frac{r}{a}\right)^8\right.$$
$$\left. + \frac{(418+33\nu)}{840}\left(\frac{r}{a}\right)^{10} - \frac{(1339+91\nu)}{13440}\left(\frac{r}{a}\right)^{12}\right\}\left(\frac{a}{R}\right)^2$$

$$\frac{u}{R} = -30f_4\cos 2\theta\left\{\frac{1}{30}\left(\frac{r}{a}\right)^3 - \frac{(19+3\nu)}{192}\left(\frac{r}{a}\right)^5 + \frac{(17+2\nu)}{120}\left(\frac{r}{a}\right)^7 - \frac{(53+3\nu)}{480}\left(\frac{r}{a}\right)^9\right.$$
$$\left. + \frac{(38+3\nu)}{840}\left(\frac{r}{a}\right)^{11} - \frac{(103+7\nu)}{13440}\left(\frac{r}{a}\right)^{13}\right\}\left(\frac{a}{R}\right)^3$$

$$\frac{u}{r} = -30f_4\cos 2\theta\left\{\frac{1}{30}\left(\frac{r}{a}\right)^2 - \frac{(19+3\nu)}{192}\left(\frac{r}{a}\right)^4 + \frac{(17+2\nu)}{120}\left(\frac{r}{a}\right)^6 - \frac{(53+3\nu)}{480}\left(\frac{r}{a}\right)^8\right.$$
$$\left. + \frac{(38+3\nu)}{840}\left(\frac{r}{a}\right)^{10} - \frac{(103+7\nu)}{13440}\left(\frac{r}{a}\right)^{12}\right\}\left(\frac{a}{R}\right)^2$$

$$\frac{1}{E}\left(\widehat{\theta\theta} - \widehat{rr}\right) = -30f_4\cos 2\theta\left\{\frac{(15-\nu)}{192}\left(\frac{r}{a}\right)^4 - \frac{(14-\nu)}{120}\left(\frac{r}{a}\right)^6 + \frac{(15-\nu)}{160}\left(\frac{r}{a}\right)^8 - \frac{(33-2\nu)}{840}\left(\frac{r}{a}\right)^{10}\right.$$
$$\left. + \frac{(91-5\nu)}{13440}\left(\frac{r}{a}\right)^{12}\right\}\left(\frac{a}{R}\right)^2$$

$$\frac{\partial v}{r\partial\theta} = -30f_4\cos 2\theta\left\{-\frac{1}{30}\left(\frac{r}{a}\right)^2 + \frac{(34+2\nu)}{192}\left(\frac{r}{a}\right)^4 - \frac{(31+\nu)}{120}\left(\frac{r}{a}\right)^6 + \frac{(49+\nu)}{240}\left(\frac{r}{a}\right)^8\right.$$
$$\left. - \frac{(71+\nu)}{840}\left(\frac{r}{a}\right)^{10} + \frac{(97+\nu)}{6720}\left(\frac{r}{a}\right)^{12}\right\}\left(\frac{a}{R}\right)^2$$

e)

$$\frac{\tau}{h} = -30 f_4 \sin 2\theta \left\{ -\frac{1}{60}\left(\frac{r}{a}\right)^3 + \frac{(17+\nu)}{192}\left(\frac{r}{a}\right)^5 - \frac{(31+\nu)}{240}\left(\frac{r}{a}\right)^7 + \frac{(49+\nu)}{480}\left(\frac{r}{a}\right)^9 \right.$$

$$\left. - \frac{(71+\nu)}{1680}\left(\frac{r}{a}\right)^{11} + \frac{(97+\nu)}{13440}\left(\frac{r}{a}\right)^{13} \right\} \left(\frac{a}{h}\right)^2$$

$$\frac{1}{2} - 6q_2 - 4s_2 = \eta\{-2p_2 - 0.03348214 f_4\}$$

$$6q_2 - \frac{1}{2} = \eta\{+2p_2 + 12r_2 - 0.27455357 f_4\}$$

$$-\frac{1}{2} - 6q_2 - 2s_2 = \eta\{+2p_2 + 6r_2 - 0.68080357 f_4\}$$

$$\frac{1}{2}(1+\nu) + 2(1+\nu)q_2 + 4s_2 = \eta\{-2(1+\nu)p_2 - 4r_2 - 0.044797642 f_4\}$$

$$2(1+\nu)q_2 - \frac{1}{2}(1+\nu) = \eta\{+2(1+\nu)p_2 + 2(3+\nu)r_2 - 0.31584822 f_4\}$$

$$q_2 + 0.6666667 s_2 - 0.0833333 = \eta\{+0.333333 p_2 + 0 \quad + 0.005580356 f_4\}$$

$$q_2 \qquad - 0.0833333 = \eta\{+0.333333 p_2 + 2r_2 - 0.045758928 f_4\}$$

$$q_2 + 0.333333 s_2 + 0.083333 = \eta\{-0.33333 p_2 - r_2 + 0.113467262 f_4\}$$

$$q_2 + 1.53846154 s_2 + 0.250000 = \eta\{-p_2 - 0.461538462 r_2 - 0.01722870 f_4\}$$

$$q_2 \qquad - 0.250000 = \eta\{+p_2 + 2.53846154 r_2 - 0.1214008 f_4\}$$

f)

$$0.66666667\, s_2 = \eta\left\{ 0 - 2r_2 + 0.05133929\, f_4 \right\}$$

$$0.3333333\, s_2 + 0.1666667 = \eta\left\{ -0.6666667\, p_2 - 3r_2 + 0.15922619\, f_4 \right\}$$

$$1.20512821\, s_2 + 0.166667 = \eta\left\{ -0.666667\, p_2 + 0.53846154\, r_2 - 0.13069597\, f_4 \right\}$$

$$1.53846154\, s_2 + 0.50000 = \eta\left\{ -2p_2 - 3r_2 + 0.10425137\, f_4 \right\}$$

$$s_2 + 0 = \eta\left\{ 0 - 3r_2 + 0.077008935\, f_4 \right\}$$

$$s_2 + 0.5000 = \eta\left\{ -2p_2 - 9r_2 + 0.47767857\, f_4 \right\}$$

$$s_2 + 0.13829787 = \eta\left\{ -0.55319149\, p_2 + 0.44680851\, r_2 - 0.10844985\, f_4 \right\}$$

$$s_2 + 0.325000 = \eta\left\{ -1.30\, p_2 - 1.95\, r_2 + 0.06776339\, f_4 \right\}$$

$$0.500000000 = \eta\left\{ -2p_2 - 6r_2 + 0.40066963\, f_4 \right\}$$

$$0.36170213 = \eta\left\{ -1.44680851\, p_2 - 9.44680851\, r_2 + 0.58612842\, f_4 \right\}$$

$$0.18670213 = \eta\left\{ -0.74680851\, p_2 - 2.39680851\, r_2 + 0.17621324\, f_4 \right\}$$

$$0.2500000 = \eta\left\{ -p_2 - 3r_2 + 0.20033482\, f_4 \right\}$$

$$0.250000 = \eta\left\{ -p_2 - 6.52941175\, r_2 + 0.40511818\, f_4 \right\}$$

$$0.25000 = \eta\left\{ -p_2 - 3.20940171\, r_2 + 0.23595505\, f_4 \right\}$$

$3.52941175\, r_2 = 0.20478336\, f_4$ $\qquad$ $0.17406585 6\, f_4$ 8)

$3.32001005\, r_2 = 0.16916313\, f_4$

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$$\frac{w}{R}=\frac{1}{2}\left(\frac{a}{R}\right)^2\left[1-\left(\frac{x}{a}\right)^2-\frac{f}{4}\left(1+\cos\frac{\pi x}{a}\right)\left(1+\cos\frac{\lambda\pi y}{a}\right)\right]$$

$$\frac{w_0}{R}=\frac{1}{2}\left(\frac{a}{R}\right)^2\left[1-\left(\frac{x}{a}\right)^2\right]$$

$$R\frac{\partial^2 w}{\partial x^2}=\left[-1+\frac{f}{8}\pi^2\cos\frac{\pi x}{a}\left(1+\cos\frac{\pi\lambda y}{a}\right)\right] \qquad R\frac{\partial^2 w_0}{\partial x^2}=-1$$

$$R\frac{\partial^2 w}{\partial y^2}=\left[+\frac{f}{8}\lambda^2\pi^2\left(1+\cos\frac{\pi x}{a}\right)\cos\frac{\pi\lambda y}{a}\right]$$

$$R\frac{\partial^2 w}{\partial x\partial y}=\left[-\frac{f}{8}\lambda\pi^2\sin\frac{\pi x}{a}\sin\frac{\pi\lambda y}{a}\right]$$

$$\nabla^4F=\pi^2\frac{E}{R^2}\left(\frac{f\lambda^2}{8}\right)\left[\frac{f\pi^2}{8}\frac{1}{4}\left(1-\cos\frac{2\pi x}{a}\right)\left(1-\cos\frac{2\pi\lambda y}{a}\right)\right.$$
$$+\left(1+\cos\frac{\pi x}{a}\right)\left(\cos\frac{\pi\lambda y}{a}\right)-\frac{f\pi^2}{8}\frac{1}{4}\left(1+2\cos\frac{\pi x}{a}+\cos\frac{2\pi x}{a}\right)$$
$$\left.\left(1+2\cos\frac{\pi\lambda y}{a}+\cos\frac{2\pi\lambda y}{a}\right)\right]$$

$$=\frac{E}{R^2}\left(\frac{f\lambda^2}{8}\right)\pi^2\left[\cos\frac{\pi\lambda y}{a}+\cos\frac{\pi x}{a}\cos\frac{\pi\lambda y}{a}\right.$$
$$+\frac{f\pi^2}{32}\left(\cancel{1}-2\cos\frac{2\pi x}{a}-2\cos\frac{2\pi\lambda y}{a}+\cancel{\cos\frac{2\pi x}{a}\cos\frac{2\pi\lambda y}{a}}\right.$$
$$-\cancel{1}-2\cos\frac{\pi x}{a}-\cancel{\cos\frac{2\pi x}{a}}-2\cos\frac{\pi\lambda y}{a}-4\cos\frac{\pi x}{a}\cos\frac{\pi\lambda y}{a}-2\cos\frac{2\pi x}{a}\cos\frac{\pi\lambda y}{a}$$
$$\left.\left.-\cancel{\cos\frac{2\pi\lambda y}{a}}-2\cos\frac{\pi x}{a}\cos\frac{2\pi\lambda y}{a}-\cancel{\cos\frac{2\pi x}{a}\cos\frac{2\pi\lambda y}{a}}\right)\right]$$

$$=\pi^2\frac{E}{R^2}\left(\frac{f\lambda^2}{8}\right)\left[-\frac{f\pi^2}{16}\cos\frac{\pi x}{a}+\left(1-\frac{f\pi^2}{16}\right)\cos\frac{\pi\lambda y}{a}+\left(1-\frac{f\pi^2}{8}\right)\cos\frac{\pi x}{a}\cos\frac{\pi\lambda y}{a}\right.$$
$$\left.-\frac{f\pi^2}{16}\left[\cos\frac{2\pi x}{a}+\cos\frac{2\pi\lambda y}{a}+\cos\frac{2\pi x}{a}\cos\frac{\pi\lambda y}{a}+\cos\frac{\pi x}{a}\cos\frac{2\pi\lambda y}{a}\right]\right]$$

$$F = E\left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\Bigg[-\frac{f\pi^2}{16}\,\frac{\cos\frac{\pi x}{a}}{\left(\frac{\pi}{a}\right)^2} + \frac{1}{\lambda^4}\left(1-\frac{f\pi^2}{16}\right)\frac{\cos\frac{\pi\lambda y}{a}}{\left(\frac{\pi}{a}\right)^2} + \frac{1}{(1+\lambda^2)^2}\left(1-\frac{f\pi^2}{8}\right)\frac{\cos\frac{\pi x}{a}\cos\frac{\pi\lambda y}{a}}{\left(\frac{\pi}{a}\right)^2}$$

$$-\frac{f\pi^2}{16}\left\{\frac{1}{16}\,\frac{\cos\frac{2\pi x}{a}}{\left(\frac{\pi}{a}\right)^2} + \frac{1}{16\lambda^4}\,\frac{\cos\frac{2\pi\lambda y}{a}}{\left(\frac{\pi}{a}\right)^2} + \frac{1}{(1+4\lambda^2)^2}\,\frac{\cos\frac{\pi x}{a}\cos\frac{2\pi\lambda y}{a}}{\left(\frac{\pi}{a}\right)^2} + \frac{1}{(4+\lambda^2)^2}\,\frac{\cos\frac{2\pi x}{a}\cos\frac{\pi\lambda y}{a}}{\left(\frac{\pi}{a}\right)^2}\right\}$$

$$+\frac{1}{\left(\frac{\pi}{a}\right)^2}\left\{a_1\cosh\frac{\pi\lambda y}{a} + b_1\left(\frac{\lambda\pi y}{a}\right)\sinh\frac{\lambda\pi y}{a}\right\}\cos\frac{\pi x}{a}$$

$$+\frac{1}{\left(\frac{2\pi}{a}\right)^2}\left\{a_2\cosh\left(\frac{2\lambda\pi y}{a}\right) + b_2\left(\frac{2\lambda\pi y}{a}\right)\sinh\frac{2\pi\lambda y}{a}\right\}\cos\frac{2\pi x}{a}\Bigg] + \frac{\sigma}{2}x^2$$

$$\frac{\sigma_y}{E} = \left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\Bigg[-\left\{a_1\cosh\frac{\pi\lambda y}{a} + b_1\left(\frac{\pi\lambda y}{a}\right)\sinh\frac{\pi\lambda y}{a}\right\}\cos\frac{\pi x}{a} - \left\{a_2\cosh\frac{2\pi\lambda y}{a} + b_2\left(\frac{2\pi\lambda y}{a}\right)\sinh\frac{2\pi\lambda y}{a}\right\}\cos\frac{2\pi x}{a}$$

$$+\frac{f\pi^2}{16}\cos\frac{\pi x}{a} + \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi x}{a}\cos\frac{\pi\lambda y}{a} + \frac{f\pi^2}{16}\left\{\frac{1}{4}\cos\frac{2\pi x}{a} + \frac{1}{(1+4\lambda^2)^2}\cos\frac{\pi x}{a}\cos\frac{2\pi\lambda y}{a}\right.$$

$$\left.+\frac{4}{(4+\lambda^2)^2}\cos\frac{2\pi x}{a}\cos\frac{\pi\lambda y}{a}\right\}\Bigg] + \frac{\sigma}{E}$$

$$a_1\cosh\pi + \pi b_1\sinh\pi = \frac{f\pi^2}{16} + \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right) + \frac{1}{(1+\lambda^2)^2}\frac{f\pi^2}{16}$$

$$(a_1+b_1)\sinh\pi + \pi b_1\cosh\pi = 0$$

$$a_2\cosh 2\pi + 2\pi b_2\sinh 2\pi = \frac{f\pi^2}{16}\left[\frac{1}{4} + \frac{4}{(4+\lambda^2)^2}\right]$$

$$(a_2+b_2)\sinh 2\pi + 2\pi b_2\cosh 2\pi = 0$$

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$$\frac{\sigma_y}{E}=\left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\Bigg[\left\{\frac{f\pi^2}{16}+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi\lambda y}{a}+\frac{1}{(1+4\lambda^2)^2}\frac{f\pi^2}{16}\cos\frac{2\pi\lambda y}{a}-a_1\cosh\frac{\pi\lambda y}{a}-b_1\left(\frac{\pi\lambda y}{a}\right)\sinh\frac{\pi\lambda y}{a}\right\}\cos\frac{\pi x}{a}$$

$$+\left\{\frac{f\pi^2}{16}\frac{1}{4}+\frac{f\pi^2}{16}\frac{4}{(4+\lambda^2)^2}\cos\frac{\pi\lambda y}{a}-a_2\cosh\frac{2\pi\lambda y}{a}-b_2\left(\frac{2\pi\lambda y}{a}\right)\sinh\frac{2\pi\lambda y}{a}\right\}\cos\frac{2\pi x}{a}\Bigg]-\frac{\sigma}{E}$$

$$\frac{1}{aE^2}\int_0^a\sigma_y^2\,dx=\frac{1}{2}\left(\frac{a}{R}\right)^4\left(\frac{f\lambda^2}{8}\right)^2\Bigg[\left\{\frac{f\pi^2}{16}+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi y}{b}+\frac{1}{(1+4\lambda^2)^2}\frac{f\pi^2}{16}\cos\frac{2\pi y}{b}\right\}-\left\{a_1\cosh\frac{\pi y}{b}+b_1\left(\frac{\pi y}{b}\right)\sinh\frac{\pi y}{b}\right\}\Bigg]^2$$

$$+\frac{1}{2}\left(\frac{a}{R}\right)^4\left(\frac{f\lambda^2}{8}\right)^2\Bigg[\left\{\frac{f\pi^2}{16}\frac{1}{4}+\frac{f\pi^2}{16}\frac{4}{(4+\lambda^2)^2}\cos\frac{\pi y}{b}\right\}-\left\{a_2\cosh\frac{2\pi y}{b}+b_2\left(\frac{2\pi y}{b}\right)\sinh\frac{2\pi y}{b}\right\}\Bigg]^2+\left(\frac{\sigma}{E}\right)^2$$

$$\frac{1}{abE^2}\int_0^a\!\!\int_0^b\sigma_y^2\,dx\,dy=\left(\frac{a}{R}\right)^4\left(\frac{f\lambda^2}{8}\right)^2\Bigg[\frac{1}{2}\left(\frac{f\pi^2}{16}\right)^2+\frac{1}{4}\frac{1}{(1+\lambda^2)^4}\left(\frac{f\pi^2}{8}-1\right)^2+\frac{1}{4}\frac{1}{(1+4\lambda^2)^4}\left(\frac{f\pi^2}{16}\right)^2+\frac{1}{2}\left(\frac{f\pi^2}{64}\right)^2+\frac{1}{4}\frac{1}{(4+\lambda^2)^4}\left(\frac{f\pi^2}{4}\right)^2\Bigg]$$

$$-\left(\frac{a}{R}\right)^4\left(\frac{f\lambda^2}{8}\right)^2\Bigg[\frac{f\pi^2}{16}\frac{1}{\pi}\int_0^\pi(a_1\cosh u+b_1u\sinh u)\,du+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\frac{1}{\pi}\int_0^\pi\cos u\,(a_1\cosh u+b_1u\sinh u)\,du$$

$$+\frac{1}{(1+4\lambda^2)^2}\frac{f\pi^2}{16}\frac{1}{\pi}\int_0^\pi\cos 2u\,(a_1\cosh u+b_1u\sinh u)\,du+\frac{f\pi^2}{64}\frac{1}{\pi}\int_0^\pi(a_2\cosh 2u+b_2(2u)\sinh 2u)\,du$$

$$+\frac{f\pi^2}{4(4+\lambda^2)^2}\frac{1}{\pi}\int_0^\pi\cos u\,(a_2\cosh 2u+b_2\,2u\sinh 2u)\,du\Bigg]$$

$$+\left(\frac{a}{R}\right)^4\left(\frac{f\lambda^2}{8}\right)^2\Bigg[\frac{1}{2}\frac{1}{\pi}\int_0^\pi(a_1\cosh u+b_1u\sinh u)^2\,du+\frac{1}{2\pi}\int_0^\pi(a_2\cosh 2u+b_1\,2u\sinh 2u)^2\,du\Bigg]$$

We have the following integrals:

$$\frac{1}{\pi}\int_0^\pi (\alpha \cosh u + \beta u \sinh u)\,du = (\alpha-\beta)\frac{\sinh\pi}{\pi} + \beta\cosh\pi$$

$$\frac{1}{\pi}\int_0^\pi \cos u(\alpha\cosh u + \beta u\sinh u)\,du = -\alpha\,\frac{\sinh\pi}{\pi} - \beta\cosh\pi$$

$$\frac{1}{\pi}\int_0^\pi \cos 2u(\alpha\cosh u + \beta u\sinh u)\,du = \frac{\alpha}{5}\,\frac{\sinh\pi}{\pi} + \beta\left(\frac{1}{5}\cosh\pi + \frac{3}{25}\,\frac{\sinh\pi}{\pi}\right)$$

$$\frac{1}{\pi}\int_0^\pi (\gamma\cosh 2u + \delta\, 2u\sinh 2u)\,du = (\gamma-\delta)\,\frac{\sinh 2\pi}{2\pi} + \delta\cosh 2\pi$$

$$\frac{1}{\pi}\int_0^\pi \cos u\,(\gamma\cosh 2u + \delta\, 2u\,\sinh 2u)\,du = \frac{4}{5}\gamma\,\frac{\sinh 2\pi}{2\pi} + \delta\left(\frac{4}{5}\cosh 2\pi + \frac{12}{25}\,\frac{\sinh 2\pi}{2\pi}\right)$$

$$\frac{1}{\pi}\int_0^\pi \left\{\alpha\cosh u + \beta u\sinh u\right\}^2 du$$

$$= \frac{\alpha^2}{2}\left(\frac{\sinh 2\pi}{2\pi} + 1\right) + \frac{\alpha\beta}{2}\left(\cosh 2\pi - \frac{\sinh 2\pi}{2\pi}\right) + \frac{\beta^2}{2}\left\{\left(\frac{2\pi^2+1}{2}\right)\frac{\sinh 2\pi}{2\pi} - \frac{1}{2}\cosh 2\pi - \frac{\pi^2}{3}\right\}$$

$$\frac{1}{\pi}\int_0^\pi \left\{\gamma\cosh 2u + \delta u\sinh 2u\right\}^2 du$$

$$= \frac{\gamma^2}{2}\left(\frac{\sinh 4\pi}{4\pi} + 1\right) + \frac{\gamma\delta}{2}\left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right) + \frac{\delta^2}{2}\left\{\left(\frac{8\pi^2+1}{2}\right)\frac{\sinh 4\pi}{4\pi} - \frac{1}{2}\cosh 4\pi - \frac{4\pi^2}{3}\right\}$$

$$\frac{1}{\pi}\int_0^{\pi} \sin u \sinh u \, du = \frac{1}{2}\,\frac{\sinh\pi}{\pi}$$

$$\frac{1}{\pi}\int_0^{\pi} \sin u \cdot u \cosh u \, du = \frac{1}{2}\left(\cosh\pi - \frac{\sinh\pi}{\pi}\right)$$

$$\frac{1}{\pi}\int_0^{\pi} \sin 2u \cdot \sinh u \, du = -\frac{2}{5}\left(\frac{\sinh\pi}{\pi}\right)$$

$$\frac{1}{\pi}\int_0^{\pi} \sin u \sinh 2u \, du = \frac{2}{5}\left(\frac{\sinh 2\pi}{2\pi}\right)$$

$$\frac{1}{\pi}\int_0^{\pi} 2u \cdot \sin u \cosh 2u \, du = \frac{2}{5}\cosh 2\pi - \frac{16}{25}\,\frac{\sinh 2\pi}{2\pi}$$

$$\frac{1}{\pi}\int_0^{\pi} u \cdot \sin 2u \cosh u \, du = -\frac{2}{5}\cosh\pi + \frac{4}{25\pi}\,\frac{\sinh\pi}{\pi}$$

$$\frac{1}{\pi}\int_0^{\pi} (\alpha \sinh u + \beta u \cosh u)^2 \, du$$

$$= \frac{\alpha^2}{2}\left(\frac{\sinh 2\pi}{2\pi} - 1\right) + \frac{\alpha\beta}{2}\left(\cosh 2\pi - \frac{\sinh 2\pi}{2\pi}\right) + \frac{\beta^2}{2}\left(\frac{2\pi^2+1}{2}\,\frac{\sinh 2\pi}{2\pi} - \frac{1}{2}\cosh 2\pi + \frac{\pi^2}{3}\right)$$

$$\frac{1}{\pi}\int_0^{\pi} (\gamma \sinh 2u + \delta u \cosh 2u)^2 \, du$$

$$= \frac{\gamma^2}{2}\left(\frac{\sinh 4\pi}{4\pi} - 1\right) + \frac{\gamma\delta}{2}\left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right) + \frac{\delta^2}{2}\left(\frac{8\pi^2+1}{2}\,\frac{\sinh 4\pi}{4\pi} - \frac{1}{2}\cosh 4\pi + \frac{4\pi^2}{3}\right)$$

Therefore

$$\frac{1}{abE^2}\int_0^a\int_0^b \sigma_y^2\,dx\,dy = \left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)^2\left[\frac{1}{2}\left(\frac{f\pi^2}{16}\right)^2 + \frac{1}{4(1+\lambda^2)^4}\left(\frac{f\pi^2}{8}-1\right)^2 + \frac{1}{4(1+4\lambda^2)^4}\left(\frac{f\pi^2}{16}\right)^2 + \frac{1}{2}\left(\frac{f\pi^2}{64}\right)^2 + \frac{1}{4(4+\lambda^2)^4}\left(\frac{f\pi^2}{4}\right)^2\right.$$

$$- \left\{ \frac{f\pi^2}{16}\left[(a_1-b_1)\frac{\sinh\pi}{\pi} + b_1\cosh\pi\right] + \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\left[-a_1\frac{\sinh\pi}{\pi} - b_1\cosh\pi\right]\right.$$

$$+ \frac{1}{(1+4\lambda^2)^2}\frac{f\pi^2}{16}\left[a_1\frac{1}{5}\frac{\sinh\pi}{\pi} + b_1\left(\frac{1}{5}\cosh\pi + \frac{3}{25}\frac{\sinh\pi}{\pi}\right)\right] + \frac{f\pi^2}{64}\left[(a_2-b_2)\frac{\sinh 2\pi}{2\pi} + b_2\cosh 2\pi\right]$$

$$\left. + \frac{(f\pi^2)}{4(4+\lambda^2)^2}\left[-\frac{4}{5}a_2\frac{\sinh 2\pi}{2\pi} + b_2\left(-\frac{4}{5}\cosh 2\pi + \frac{12}{25}\frac{\sinh 2\pi}{2\pi}\right)\right]\right\}$$

$$+ \frac{1}{4}a_1^2\left(\frac{\sinh 2\pi}{2\pi}+1\right) + \frac{a_1b_1}{4}\left(\cosh 2\pi - \frac{\sinh 2\pi}{2\pi}\right) + \frac{b_1^2}{4}\left(\frac{2\pi^2+1}{2}\frac{\sinh 2\pi}{2\pi} - \frac{1}{2}\cosh 2\pi - \frac{\pi^2}{3}\right)$$

$$\left. + \frac{1}{4}a_2^2\left(\frac{\sinh 4\pi}{4\pi}+1\right) + \frac{a_2b_2}{4}\left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right) + \frac{b_2^2}{4}\left(\frac{8\pi^2+1}{2}\frac{\sinh 4\pi}{4\pi} - \frac{1}{2}\cosh 4\pi - \frac{4\pi^2}{3}\right)\right]$$

$$\frac{\sigma_x}{E} = \lambda^2\left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\Bigg[\left\{(a_1+2b_1)\cosh\frac{\pi\lambda y}{a} + b_1\left(\frac{\pi\lambda y}{a}\right)\sinh\frac{\pi\lambda y}{a}\right\}\cos\frac{\pi x}{a} + \left\{(a_2+2b_2)\cosh\frac{2\pi\lambda y}{a} + b_2\left(\frac{2\pi\lambda y}{a}\right)\sinh\frac{2\pi\lambda y}{a}\right\}\cos\frac{2\pi x}{a}$$

$$+ \frac{1}{\lambda^4}\left(\frac{f\pi^2}{16}-1\right)\cos\frac{\pi\lambda y}{a} + \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi x}{a}\cos\frac{\pi\lambda y}{a} + \frac{f\pi^2}{16}\left\{\frac{1}{4\lambda^4}\cos\frac{2\pi\lambda y}{a} + \frac{4}{(1+4\lambda^2)^2}\cos\frac{\pi x}{a}\cos\frac{2\pi\lambda y}{a}\right.$$

$$\left. + \frac{1}{(4+\lambda^2)^2}\cos\frac{2\pi x}{a}\cos\frac{\pi\lambda y}{a}\right\}\Bigg]$$

$$= \lambda^2\left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\Bigg[\left\{\frac{1}{\lambda^4}\left(\frac{f\pi^2}{16}-1\right)\cos\frac{\pi\lambda y}{a} + \frac{f\pi^2}{64\lambda^4}\cos\frac{2\pi\lambda y}{a}\right\}$$

$$+ \left\{\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi\lambda y}{a} + \frac{f\pi^2}{4(1+4\lambda^2)^2}\cos\frac{2\pi\lambda y}{a} + (a_1+2b_1)\cosh\frac{\pi\lambda y}{a} + b_1\left(\frac{\pi\lambda y}{a}\right)\sinh\frac{\pi\lambda y}{a}\right\}\cos\frac{\pi x}{a}$$

$$+ \left\{\frac{f\pi^2}{16(4+\lambda^2)^2}\cos\frac{\pi\lambda y}{a} + (a_2+2b_2)\cosh\frac{2\pi\lambda y}{a} + b_2\left(\frac{2\pi\lambda y}{a}\right)\sinh\frac{2\pi\lambda y}{a}\right\}\cos\frac{2\pi x}{a}\Bigg]$$

$$
\begin{aligned}
\frac{1}{abE^2}\int_0^a\!\!\int_0^b \sigma_x^2\,dx\,dy = \lambda^4\left(\frac{a}{R}\right)^4\left(\frac{f\lambda^2}{8}\right)^2\Bigg[& \frac{1}{2}\frac{1}{\lambda^8}\left(\frac{f\pi^2}{16}-1\right)^2+\frac{1}{2}\frac{1}{\lambda^8}\left(\frac{f\pi^2}{64}\right)^2+\frac{1}{4}\frac{1}{(1+\lambda^2)^4}\left(\frac{f\pi^2}{8}-1\right)^2+\frac{1}{4(1+4\lambda^2)^4}\left(\frac{f\pi^2}{4}\right)^2 \\
& +\frac{1}{4(4+\lambda^2)^4}\left(\frac{f\pi^2}{16}\right)^2 \\
& +\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\left\{-(a_1+2b_1)\frac{\sinh\pi}{\pi}-b_1\cosh\pi\right\}+\frac{f\pi^2}{4(1+4\lambda^2)^2}\left\{\frac{a_1+2b_1}{5}\frac{\sinh\pi}{\pi}+b_1\left(\frac{1}{5}\cosh\pi+\frac{3}{25}\frac{\sinh\pi}{\pi}\right)\right\} \\
& +\frac{f\pi^2}{16(4+\lambda^2)^2}\left\{-\frac{4}{5}(a_2+2b_2)\frac{\sinh 2\pi}{2\pi}+b_2\left(-\frac{4}{5}\cosh 2\pi+\frac{12}{25}\frac{\sinh 2\pi}{2\pi}\right)\right\} \\
& +\frac{1}{4}(a_1+2b_1)^2\left(\frac{\sinh 2\pi}{2\pi}+1\right)+\frac{(a_1+2b_1)b_1}{4}\left(\cosh 2\pi-\frac{\sinh 2\pi}{2\pi}\right)+\frac{b_1^2}{4}\left(\frac{2\pi^2+1}{2}\frac{\sinh 2\pi}{2\pi}-\frac{1}{2}\cosh 2\pi-\frac{\pi^2}{3}\right) \\
& +\frac{1}{4}(a_2+2b_2)^2\left(\frac{\sinh 4\pi}{4\pi}+1\right)+\frac{(a_2+2b_2)b_2}{4}\left(\cosh 4\pi-\frac{\sinh 4\pi}{4\pi}\right)+\frac{b_2^2}{4}\left(\frac{8\pi^2+1}{2}\frac{\sinh 4\pi}{4\pi}-\frac{1}{2}\cosh 4\pi-\frac{4\pi^2}{3}\right)\Bigg]
\end{aligned}
$$

$$\frac{\tau_{xy}}{E} = \lambda\left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\Bigg[\left\{(a_1+b_1)\sinh\frac{\pi\lambda y}{a} + b_1\left(\frac{\lambda\pi y}{a}\right)\cosh\frac{\lambda\pi y}{a}\right\}\sin\frac{\pi x}{a} + \left\{(a_2+b_2)\sinh\frac{2\pi\lambda y}{a} + b_2\left(\frac{2\pi\lambda y}{a}\right)\cosh\frac{2\pi\lambda y}{a}\right\}\sin\frac{2\pi x}{a}$$

$$+ \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\sin\frac{\pi x}{a}\sin\frac{\pi\lambda y}{a} + \frac{-f\pi^2}{8}\frac{1}{(1+4\lambda^2)^2}\sin\frac{\pi x}{a}\sin\frac{2\pi\lambda y}{a} + \frac{f\pi^2}{8}\frac{1}{(4+\lambda^2)^2}\sin\frac{2\pi x}{a}\sin\frac{\pi\lambda y}{a}\Bigg]$$

$$= \lambda\left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\Bigg[\left\{\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\sin\frac{\pi\lambda y}{a} + \frac{1}{(1+4\lambda^2)^2}\frac{f\pi^2}{8}\sin\frac{2\pi\lambda y}{a} + (a_1+b_1)\sinh\frac{\pi\lambda y}{a} + b_1\left(\frac{\lambda\pi y}{a}\right)\cosh\frac{\lambda\pi y}{a}\right\}\sin\frac{\pi x}{a}$$

$$+ \left\{\frac{1}{(4+\lambda^2)^2}\left(\frac{f\pi^2}{8}\right)\sin\frac{\pi\lambda y}{a} + (a_2+b_2)\sinh\frac{2\pi\lambda y}{a} + b_2\left(\frac{2\pi\lambda y}{a}\right)\cosh\frac{2\pi\lambda y}{a}\right\}\sin\frac{2\pi x}{a}\Bigg]$$

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$$\frac{1}{abE^2}\int_0^a\int_0^b \tau_{xy}^2\,dy\,dx = \lambda^2\left(\frac{a}{R}\right)^4\left(\frac{f\lambda^2}{8}\right)^2\Bigg[\frac{1}{4(1+\lambda^2)^4}\left(\frac{f\pi^2}{8}-1\right)^2+\frac{1}{4(1+4\lambda^2)^4}\left(\frac{f\pi^2}{8}\right)^2+\frac{1}{4(4+\lambda^2)^2}\left(\frac{f\pi^2}{8}\right)^2$$

$$+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\left\{\frac{1}{2}(a_1+b_1)\frac{\sinh\pi}{\pi}+\frac{b_1}{2}\left(\cosh\pi-\cancel{\frac{\sinh\pi}{\pi}}\right)\right\}+\frac{1}{(1+4\lambda^2)^2}\frac{f\pi^2}{8}\left\{-\frac{2}{5}(a_1+b_1)\frac{\sinh\pi}{\pi}+b_1\left(-\frac{2}{5}\cosh\pi+\frac{4}{25\pi}\frac{\sinh\pi}{\pi}\right)\right\}$$

$$+\frac{1}{(4+\lambda^2)^2}\left(\frac{f\pi^2}{8}\right)\left\{\frac{2}{5}(a_2+b_2)\frac{\sinh 2\pi}{2\pi}+b_2\left(\frac{2}{5}\cosh 2\pi-\frac{16}{25}\frac{\sinh 2\pi}{2\pi}\right)\right\}$$

$$+\frac{(a_1+b_1)^2}{4}\left(\frac{\sinh 2\pi}{2\pi}-1\right)+\frac{(a_1+b_1)b_1}{4}\left(\cosh 2\pi-\frac{\sinh 2\pi}{2\pi}\right)+\frac{b_1^2}{4}\left(\frac{2\pi^2+1}{2}\frac{\sinh 2\pi}{2\pi}-\frac{1}{2}\cosh 2\pi+\frac{\pi^2}{3}\right)$$

$$+\frac{(a_2+b_2)^2}{4}\left(\frac{\sinh 4\pi}{4\pi}-1\right)+\frac{(a_2+b_2)b_2}{4}\left(\cosh 4\pi-\frac{\sinh 4\pi}{4\pi}\right)+\frac{b_2^2}{4}\left(\frac{8\pi^2+1}{2}\frac{\sinh 4\pi}{4\pi}-\frac{1}{2}\cosh 4\pi+\frac{4\pi^2}{3}\right)\Bigg]$$

$$F_1 = (f\pi^2)^2 \Big\{ \frac{1}{512}\frac{1}{\lambda^4} + \frac{1}{8192}\frac{1}{\lambda^4} + \frac{1}{256}\frac{\lambda^4}{(1+\lambda^2)^4} + \frac{\lambda^4}{64(1+4\lambda^2)^4} + \frac{\lambda^4}{1024(4+\lambda^2)^4}$$

$$+ \frac{1}{512} + \frac{1}{8192} + \frac{1}{256}\frac{1}{(1+\lambda^2)^4} + \frac{1}{1024(1+4\lambda^2)^4} + \frac{1}{64(4+\lambda^2)^2} + \frac{2\lambda^2}{256(1+\lambda^2)^4} + \frac{2\lambda^2}{256(1+4\lambda^2)}$$

$$+ \frac{2\lambda^2}{256(4+\lambda^2)^2} \Big\}$$

$$- (f\pi^2)\Big\{ \frac{1}{16}\frac{1}{\lambda^4} + \frac{\lambda^4}{16(1+\lambda^2)^4} + \frac{1}{16(1+\lambda^2)^4} + \frac{2\lambda^2}{16(1+\lambda^2)^2} \Big\} + \Big\{ \frac{1}{2\lambda^4} + \frac{\lambda^4}{4(1+\lambda^2)^4} + \frac{1}{4(1+\lambda^2)^4} + \frac{2\lambda^2}{4(1+\lambda^2)^4} \Big\}$$

$$= (f\pi^2)^2 \Big\{ \frac{17}{8192}\Big(1+\frac{1}{\lambda^4}\Big) + \frac{1}{256}\frac{1}{(1+\lambda^2)^2} + \frac{1}{1024(1+4\lambda^2)^2} + \frac{1}{1024(4+\lambda^2)^2} \Big\}$$

$$- (f\pi^2)\Big\{ \frac{1}{16\lambda^4} + \frac{1}{16(1+\lambda^2)^2} \Big\} + \Big\{ \frac{1}{2\lambda^4} + \frac{1}{4(1+\lambda^2)^2} \Big\}$$

$a_1 \quad \dfrac{\sinh \pi}{\pi}$

$$-\frac{\lambda^4}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)+\frac{\lambda^4}{20(1+4\lambda^2)^2}(f\pi^2)-\frac{f\pi^2}{16}+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)$$

$$-\frac{1}{80(1+4\lambda^2)^2}(f\pi^2)+\frac{\lambda^2}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)-\frac{\lambda^2(f\pi^2)}{20(1+4\lambda^2)^2}=g_1(\lambda)$$

$b_1 \quad \dfrac{\sinh \pi}{\pi}$

$$-\frac{2\lambda^4}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)+\frac{13}{25}\frac{\lambda^4}{4(1+4\lambda^2)^2}(f\pi^2)+\frac{f\pi^2}{16}-\frac{3}{25}\frac{1}{16(1+4\lambda^2)^2}(f\pi^2)$$

$$-\frac{3}{25}\frac{\lambda^2 f\pi^2}{4(1+4\lambda^2)^2}=g_2(\lambda)$$

$b_1 \quad \cosh \pi$

$$-\frac{\lambda^4}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)+\frac{\lambda^4 f\pi^2}{20(1+4\lambda^2)^2}-\frac{f\pi^2}{16}+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)-\frac{f\pi^2}{80(1+4\lambda^2)^2}$$

$$+\frac{\lambda^2}{(1+\lambda^2)^2}-\frac{\lambda^2}{10(1+4\lambda^2)^2}(f\pi^2)=g_1(\lambda)$$

$a_2 \quad \dfrac{\sinh 2\pi}{2\pi}$

$$-\frac{\lambda^4(f\pi^2)}{20(4+\lambda^2)^2}-\frac{f\pi^2}{64}+\frac{f\pi^2}{5(4+\lambda^2)^2}+\frac{\lambda^2(f\pi^2)}{10(4+\lambda^2)^2}=g_3(\lambda)$$

$b_2 \dfrac{\sinh 2\pi}{2\pi}$

$$-\frac{7\lambda^4}{25\times 4(4+\lambda^2)^2}(f\pi^2)+\frac{f\pi^2}{64}-\frac{12}{100}\frac{(f\pi^2)}{(4+\lambda^2)^2}-\frac{3\times 2\lambda^2}{25\times 4(4+\lambda^2)^2}(f\pi^2)=g_4(\lambda)$$

$b_2 \cosh 2\pi$

$$-\frac{\lambda^4(f\pi^2)}{20(4+\lambda^2)^2}-\frac{f\pi^2}{64}+\frac{(f\pi^2)}{5(4+\lambda^2)^2}+\frac{4\lambda^2(f\pi^2)}{40(4+\lambda^2)^2}=g_3(\lambda)$$

$$\cosh\pi\, a_1+\pi\sinh\pi\, b_1=\left\{\frac{1}{16}+\frac{1}{8(1+\lambda^2)^2}+\frac{1}{16(1+4\lambda^2)^2}\right\}(f\pi^2)-\frac{1}{(1+\lambda^2)^2}$$

$$\sinh\pi\, a_1+(\sinh\pi+\pi\cosh\pi)b_1=0$$

$$b_1=-\frac{\frac{\sinh\pi}{\pi}}{\frac{\sinh 2\pi}{2\pi}+1}\left[\left\{\frac{1}{16}+\boxed{\frac{1}{8(1+\lambda^2)^2}}^{H_1(\lambda)}+\frac{1}{16(1+4\lambda^2)^2}\right\}f\pi^2-\boxed{\frac{1}{(1+\lambda^2)^2}}^{H_2(\lambda)}\right]$$

$$a_1=+\frac{\frac{\sinh\pi}{\pi}+\cosh\pi}{\frac{\sinh 2\pi}{2\pi}+1}\left[\left\{\frac{1}{16}+\frac{1}{8(1+\lambda^2)^2}+\frac{1}{16(1+4\lambda^2)^2}\right\}f\pi^2-\frac{1}{(1+\lambda^2)^2}\right]$$

$$a_2=+\frac{\frac{\sinh 2\pi}{2\pi}+\cosh 2\pi}{\frac{\sinh 4\pi}{4\pi}+1}\left[\frac{1}{64}+\boxed{\frac{1}{4(4+\lambda^2)^2}}^{H_3(\lambda)}\right]f\pi^2$$

$$b_2=-\frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi}+1}\left[\frac{1}{64}+\frac{1}{4(4+\lambda^2)^2}\right]f\pi^2$$

Thus

$$F_9 = -\left[H_1 f\pi^2 - H_2\right]\left[-\frac{\sinh\pi}{\pi}\, g_1\left(\frac{\sinh\pi}{\pi} + \cosh\pi\right)\right.$$

$$\left. + \left(\frac{\sinh\pi}{\pi}\right)^2 g_2 + \frac{\sinh\pi}{\pi}\cdot\cosh\pi\; g_1\right]\frac{1}{\frac{\sinh 2\pi}{2\pi}+1}$$

$$-\left[H_3 (f\pi^2)^2\right]\left[-g_3\frac{\sinh 2\pi}{2\pi}\left(\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi\right) + \left(\frac{\sinh 2\pi}{2\pi}\right)^2 g_4 + g_3\frac{\sinh 2\pi}{2\pi}\cosh 2\pi\right]$$

$$\cdot\frac{1}{\frac{\sinh 4\pi}{4\pi}+1}$$

$$= -\frac{\left(\frac{\sinh\pi}{\pi}\right)^2}{\frac{\sinh 2\pi}{2\pi}+1}\left[H_1 (f\pi^2) - H_2\right]\left[g_2 - g_1\right]$$

$$-\frac{\left(\frac{\sinh 2\pi}{2\pi}\right)^2}{\frac{\sinh 4\pi}{4\pi}+1} H_3 (f\pi^2)^2\left[g_4 - g_3\right]$$

$$g_2 - g_1 = -\frac{\lambda^2}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8} - 1\right) + \frac{2\lambda^2}{25(1+4\lambda^2)^2}(f\pi^2) + \frac{1}{8}(f\pi^2) - \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8} - 1\right)$$

$$+\frac{1}{200(1+4\lambda^2)^2}(f\pi^2) - \frac{\lambda^2}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8} - 1\right) + \frac{\lambda^2}{25(1+4\lambda^2)^2}(f\pi^2)$$

$$= \left(\frac{f\pi^2}{8} - 1\right)\left(\frac{\lambda^2}{(1+\lambda^2)^2} - 1\right) + \frac{1}{8}(f\pi^2) + (f\pi^2)\left(\frac{1}{200} - \frac{\lambda^2}{25(1+4\lambda^2)^2}\right)$$

$$= (f\pi^2)\left\{\frac{1}{200} + \frac{\lambda^2}{8(1+\lambda^2)^2} - \frac{\lambda^2}{25(1+4\lambda^2)^2}\right\} + \left\{1 - \frac{\lambda^2}{(1+\lambda^2)^2}\right\}$$

$$g_4 - g_3 = -\frac{\lambda^4}{50(4+\lambda^2)^2} + \frac{1}{32} - \frac{8}{25(4+\lambda^2)^2} - \frac{8\lambda^2}{50(4+\lambda^2)^2}$$

$$= \frac{1}{32} - \frac{1}{50} = \frac{9}{800}$$

$$H_1(\lambda) = \frac{1}{8}\left\{\frac{1}{2} + \frac{1}{(1+\lambda^2)^2} + \frac{1}{2(1+4\lambda^2)^2}\right\}$$

$$H_2(\lambda) = \frac{1}{(1+\lambda^2)^2}$$

$$H_3(\lambda) = \frac{1}{\text{[illegible]}}\left\{\frac{1}{16} + \frac{1}{(4+\lambda^2)^2}\right\}$$

$$H_4(\lambda) = \frac{1}{200} + \frac{\lambda^2}{8(1+\lambda^2)^2} - \frac{\lambda^2}{25(1+4\lambda^2)^2}$$

$$H_5(\lambda) = 1 - \frac{\lambda^2}{(1+\lambda^2)^2}$$

$$F_2 = -\frac{\left(\frac{\sinh \pi}{\pi}\right)^2}{\frac{\sinh 2\pi}{2\pi} + 1}\left[H_1(-\pi^2) - H_2\right]\left[H_4(-\pi^2) + H_5\right]$$

$$- \frac{9}{800}\,\frac{\left(\frac{\sinh 2\pi}{2\pi}\right)^2}{\frac{\sinh 4\pi}{4\pi} + 1}\; H_3(-\pi^2)^2$$

$$F_3 = \left[H_1(1+\pi^2) - H_2\right]^2 \times \frac{1}{\left(\frac{\sinh 2\pi}{2\pi}+1\right)^2}$$

$$\Bigg[\frac{1}{4}\left(\frac{\sinh\pi}{\pi}+\cosh\pi\right)^2\left\{(1+\lambda^2)^2\frac{\sinh 2\pi}{2\pi}+(1-\lambda^2)^2\right\}$$

$$-\frac{1}{4}\,\frac{\sinh\pi}{\pi}\left(\frac{\sinh\pi}{\pi}+\cosh\pi\right)\left\{(3\lambda^4+2\lambda^2-1)\frac{\sinh 2\pi}{2\pi}+(1+\lambda^2)^2\cosh 2\pi-4(1-\lambda^2)\lambda^2\right\}$$

$$+\frac{1}{4}\left(\frac{\sinh\pi}{\pi}\right)^2\Bigg\{\left[\pi^2(1+\lambda^2)^2+\frac{1}{2}(5\lambda^4+2\lambda^2+1)\right]\frac{\sinh 2\pi}{2\pi}+\frac{1}{2}(3\lambda^4+2\lambda^2-1)\cosh 2\pi$$

$$-\left[\frac{\pi^2}{3}(1-\lambda^2)^2+2(1-2\lambda^4)\right]\Bigg\}\Bigg]$$

$$+\ H_3^2(4\pi^2)^2\times\frac{1}{\left(\frac{\sinh 4\pi}{4\pi}+1\right)^2}$$

$$\Bigg[\frac{1}{4}\left(\frac{\sinh 2\pi}{2\pi}+\cosh 2\pi\right)^2\left\{(1+\lambda^2)^2\frac{\sinh 4\pi}{4\pi}+(1-\lambda^2)^2\right\}$$

$$-\frac{1}{4}\,\frac{\sinh 2\pi}{2\pi}\left(\frac{\sinh 2\pi}{2\pi}+\cosh 2\pi\right)\left\{(3\lambda^4+2\lambda^2-1)\frac{\sinh 4\pi}{4\pi}+(1+\lambda^2)^2\cosh 4\pi-4(1-\lambda^2)\lambda^2\right\}$$

$$+\frac{1}{4}\left(\frac{\sinh 2\pi}{2\pi}\right)^2\Bigg\{\left[4\pi^2(1+\lambda^2)^2+\frac{1}{2}(5\lambda^4+2\lambda^2+1)\right]\frac{\sinh 4\pi}{4\pi}+\frac{1}{2}(3\lambda^4+2\lambda^2-1)\cosh 2\pi$$

$$-\left[\frac{4\pi^2}{3}(1-\lambda^2)^2+2(1-2\lambda^4)\right]\Bigg\}\Bigg]$$

$$\frac{1}{12}\frac{1}{ab}t^2\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial x^2}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{8}\right)^2\frac{3}{4}$$ $\underline{\underline{467}}$

$$\frac{1}{12}\frac{1}{ab}t^2\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial y^2}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2\lambda^2}{8}\right)^2\frac{3}{4}$$

$$\frac{1}{12}\frac{1}{ab}t^2\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2\lambda}{8}\right)^2\frac{1}{4}$$ $\underline{\underline{\times 2}}$!

the bending energy $/(E\,ab)t\left(\frac{1}{2}\right)$

$$= \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f}{8}\right)^2\pi^4\left[\frac{3}{4}(1+\lambda^4)+\frac{1}{2}\lambda^2\right]$$

$$= \left(\frac{t}{R}\right)^2\left(\frac{f}{8}\right)^2\pi^4\left[\frac{1}{16}(1+\lambda^4)+\frac{1}{24}\lambda^2\right]$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{1}{2}\left[\frac{f\lambda\pi}{8}\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\lambda\pi y}{a}\right]^2\left(\frac{a}{R}\right)^2$$

$$= \left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\left[\frac{f\pi^2}{16}\frac{1}{4}\left(3+2\cos\frac{\pi x}{a}+\cos\frac{2\pi x}{a}\right)\left(1-\frac{\cos 2\pi\lambda y}{a}\right)\right]$$

$$\frac{\partial v}{\partial y} = \left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\left[-\frac{3}{4}\frac{f\pi^2}{16}+----\right] - \frac{\sigma}{E}$$

$$\left(\frac{v}{b}\right)_{y=b} = \left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\left[-\frac{3}{4}\frac{f\pi^2}{16}+---\right] - \frac{\sigma}{E}$$

$$2\times\frac{\Delta U}{Eabt} = -\frac{\sigma}{E}\left(\frac{a}{R}\right)^2\left(\frac{f\lambda}{\delta}\right)^2\frac{3}{32}(4\pi^2) - 2\left(\frac{\sigma}{E}\right)^2$$

$$= -\left(\frac{a}{R}\right)^2\left(\frac{f}{\delta}\right)^2\lambda^2\frac{3}{4}\pi^2\frac{\sigma}{E} - 2\left(\frac{\sigma}{E}\right)^2$$

---

for limiting case of $\lambda << 1$

$$\frac{3}{2}\pi^2\frac{\sigma}{E} = \frac{1}{\lambda^2}\left(\frac{a}{R}\right)^2\left\{\frac{17}{\underset{1024}{2048}}\pi^4 f^2 - \frac{3}{16}\pi^2 f + 1\right\} + \frac{1}{8}\pi^4\frac{\left(\frac{t}{R}\right)^2}{\left(\frac{a}{R}\right)^2}\frac{1}{\lambda^2}$$

$$\lambda^2 K = \rho^2\left\{\frac{17}{3072}\pi^2 f^2 - \frac{1}{8}f + \frac{2}{3\pi^2}\right\} + \frac{1}{12}\frac{\pi^2}{\rho^2}$$

$$= \frac{\pi^2}{\rho^2}\left\{\frac{17}{768}\left(\frac{\delta}{t}\right)^2 + \frac{1}{12}\right\} - \frac{1}{4}\left(\frac{\delta}{t}\right) + \frac{2}{3}\frac{\rho^2}{\pi^2}$$

$$= 2\left\{\frac{17}{1152}\left(\frac{\delta}{t}\right)^2 + \frac{1}{18}\right\} - \frac{1}{4}\left(\frac{\delta}{t}\right)$$

$$\left(\frac{1}{64} - \frac{17}{1152}\right)\left(\frac{\delta}{t}\right)^2 = \frac{1}{18} \qquad \left(\frac{\delta}{t}\right)^2 = \frac{1}{18}\times 1152 \qquad \left(\frac{\delta}{t}\right) = 8$$

$$\frac{w}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left[1-\left(\frac{x}{a}\right)^2 - \frac{f}{4}\left(1+\cos\frac{\pi x}{a}\right)\left(1+\cos\frac{\pi y}{b}\right)\right]$$ <u>469</u>

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$$\frac{w_0}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left[1-\left(\frac{x}{a}\right)^2\right]$$

$$\sigma_x = E\left\{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right\}$$

$$\sigma_y = E\left\{\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\}$$

$$\tau_{xy} = E\left\{\frac{1}{2}\frac{\partial v}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right\}$$

By using the equilibrium equation $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$, we have

$$\frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\left\{\frac{1}{2}\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y} + \frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}\right\}$$

$$= -\left\{\frac{1}{2}\frac{\partial w}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 w}{\partial y^2}\right) + \frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\right\}$$

We have

$$\frac{\partial w}{\partial x} = \left(\frac{a}{R}\right)\left[-\left(\frac{x}{a}\right) + \frac{f}{8}\pi \sin\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)\right]$$

$$\frac{\partial w}{\partial y} = \left(\frac{a}{R}\right)\left[\frac{f}{8}\pi\left(\frac{a}{b}\right)\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}\right]$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{R}\left[-1 + \frac{f}{8}\pi^2\cos\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)\right]$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{R}\left[\frac{f}{8}\pi^2\left(\frac{a}{b}\right)^2\left(1+\cos\frac{\pi x}{a}\right)\cos\frac{\pi y}{b}\right]$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{R}\left[-\frac{f}{8}\pi^2\left(\frac{a}{b}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right]$$

$$R\left(\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial y^2}\right)=-\left(\frac{a}{R}\right)\left[\frac{f}{g}\pi\left(\frac{a}{b}\right)\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}\left\{\frac{f}{g}\pi^2\cos\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)+\frac{f}{g}\pi^2\left(\frac{a}{b}\right)^2\left(1+\cos\frac{\pi x}{a}\right)\cos\frac{\pi y}{b}-1\right\}\right.$$

$$\left.-\left\{\frac{f}{g}\pi\sin\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)-\left(\frac{x}{a}\right)\right\}\frac{f}{g}\pi^2\left(\frac{a}{b}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right]$$

$$=-\left(\frac{a}{R}\right)\left[\left(\frac{f}{g}\right)^2\pi^3\left(\frac{a}{b}\right)\frac{1}{4}\left(1+2\cos\frac{\pi x}{a}+\cos\frac{2\pi x}{a}\right)\left(2\sin\frac{\pi y}{b}+\sin\frac{2\pi y}{b}\right)\right.$$

$$+\left(\frac{f}{g}\right)^2\pi^3\left(\frac{a}{b}\right)^3\frac{1}{4}\left(3+4\cos\frac{\pi x}{a}+\cos\frac{2\pi x}{a}\right)\sin\frac{2\pi y}{b}-\frac{f}{g}\pi\left(\frac{a}{b}\right)\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}$$

$$\left.-\left(\frac{f}{g}\right)^2\pi^3\left(\frac{a}{b}\right)\frac{1}{4}\left(1-\cos\frac{2\pi x}{a}\right)\left(2\sin\frac{\pi y}{b}+\sin\frac{2\pi y}{b}\right)+\frac{f}{g}\pi^2\left(\frac{a}{b}\right)\left(\frac{x}{a}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right]$$

$$=-\left(\frac{a}{R}\right)\left(\frac{f}{g}\right)\pi\left(\frac{a}{b}\right)\left[\frac{1}{4}\left(\frac{f}{g}\right)\pi^2\left(4\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}+4\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}+2\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}+2\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\right)\right.$$

$$+\frac{1}{4}\left(\frac{f}{g}\right)\pi^2\left(\frac{a}{b}\right)^2\left(3\sin\frac{2\pi y}{b}+4\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}+\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\right)$$

$$\left.-\left\{1+\cos\frac{\pi x}{a}-\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\right\}\sin\frac{\pi y}{b}\right]$$

$$=-\left(\frac{a}{R}\right)\left(\frac{f}{g}\right)\pi\left(\frac{a}{b}\right)\left[\frac{1}{4}\left(\frac{f}{g}\right)\pi^2\left\{3\left(\frac{a}{b}\right)^2\sin\frac{2\pi y}{b}+4\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}+4\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}+\left(2+4\left(\frac{a}{b}\right)^2\right)\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}\right.\right.$$

$$\left.+\left(2+\left(\frac{a}{b}\right)^2\right)\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\right\}$$

$$\left.-\left\{\sin\frac{\pi y}{b}+\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}-\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right\}\right]$$

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Consider $$R\left(\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial y^2}\right) = \left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

Let $$v = X\sin\frac{\pi y}{b}$$

$$R\left[\frac{d^2 X}{dx^2} - 2\left(\frac{\pi}{b}\right)^2 X\right] = \left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}$$

Let $$X = A\cos\frac{\pi x}{a} + B\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}$$

$$\frac{dX}{dx} = -\left(\frac{\pi}{a}\right)A\sin\frac{\pi x}{a} + \left(\frac{\pi}{a}\right)B\sin\frac{\pi x}{a} + \left(\frac{\pi}{a}\right)B\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}$$

$$\frac{d^2 X}{dx^2} = -\left(\frac{\pi}{a}\right)^2(A-2B)\cos\frac{\pi x}{a} - \left(\frac{\pi}{a}\right)^2 B\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}$$

$$R\left[\left\{-\left(\frac{\pi}{a}\right)^2(A-2B) - 2\left(\frac{\pi}{b}\right)^2 A\right\}\cos\frac{\pi x}{a} - \left\{\left(\frac{\pi}{a}\right)^2 B + 2\left(\frac{\pi}{b}\right)^2 B\right\}\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\right] = \left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}$$

$$\therefore\quad B = -\frac{1}{\left(\frac{\pi}{a}\right)^2 + 2\left(\frac{\pi}{b}\right)^2}\,\frac{1}{R}$$

$$\left[\left(\frac{\pi}{a}\right)^2 + 2\left(\frac{\pi}{b}\right)^2\right]A = 2\left(\frac{\pi}{a}\right)^2 B, \qquad A = \frac{2\left(\frac{\pi}{a}\right)^2}{\left(\frac{\pi}{a}\right)^2 + 2\left(\frac{\pi}{b}\right)^2}B = -\frac{2\left(\frac{\pi}{a}\right)^2}{\left[\left(\frac{\pi}{a}\right)^2 + 2\left(\frac{\pi}{b}\right)^2\right]^2}\frac{1}{R}$$

The particular integral:

$$\frac{v}{R} = +\left(\frac{a}{R}\right)\left(\frac{f}{g}\right)\pi\left(\frac{a}{b}\right)\frac{1}{R^2}\Bigg[\frac{1}{4}\left(\frac{f}{g}\right)\pi^2\Bigg\{\frac{3\left(\frac{a}{b}\right)^2}{2\left(\frac{2\pi}{b}\right)^2}\sin\frac{2\pi y}{b} + \frac{4}{\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{4}{\left(\frac{2\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2}\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}$$

$$+\frac{2+4\left(\frac{a}{b}\right)^2}{\left(\frac{\pi}{a}\right)^2+2\left(\frac{2\pi}{b}\right)^2}\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b} + \frac{2+\left(\frac{a}{b}\right)^2}{\left(\frac{2\pi}{a}\right)^2+2\left(\frac{2\pi}{b}\right)^2}\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\Bigg\}$$

$$-\Bigg\{\frac{1}{2\left(\frac{\pi}{b}\right)^2}\sin\frac{\pi y}{b} + \frac{1\cdot\left(\frac{\pi}{a}\right)^2}{\left[\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2\right]^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} - \cancel{\frac{2\left(\frac{\pi}{a}\right)^2}{\left[\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2\right]^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}}$$

$$-\frac{1}{\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2}\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}\Bigg\}\Bigg]\Bigg]$$

$$\frac{v}{R} = \left(\frac{a}{R}\right)^3\left(\frac{f}{g}\right)\pi\lambda\Bigg[\frac{1}{4}\left(\frac{f}{g}\right)\Bigg\{\frac{3}{8}\sin\frac{2\pi y}{b} + \frac{4}{(1+2\lambda^2)}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{4}{(4+2\lambda^2)}\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}$$

$$+\frac{2+4\lambda^2}{(1+8\lambda^2)}\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b} + \frac{2+\lambda^2}{4+8\lambda^2}\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\Bigg\}$$

$$-\frac{1}{\pi^2}\Bigg\{\frac{1}{2\lambda^2}\sin\frac{\pi y}{b} + \frac{1}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} - \frac{1}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}\Bigg\}$$

$$+\frac{1}{\pi}a_0\left(\frac{\pi y}{b}\right) + \frac{1}{\pi}a_1\cosh\frac{\sqrt{2}\lambda\pi x}{a}\sin\frac{\pi y}{b}\Bigg]$$

Add the complementary function.

$$\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial y^2} = 0, \qquad A_0\left(\frac{\pi y}{b}\right) + A_1 \cosh\frac{\sqrt{2}\,\pi x}{b} \sin\frac{\pi y}{b}$$

$$\frac{\partial v}{\partial y} = \left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right) \lambda^2 \Big[\frac{f\pi^2}{32}\Big\{ \frac{3}{4}\cos\frac{2\pi y}{b} + \frac{4}{(1+2\lambda^2)}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} + \frac{4}{(4+2\lambda^2)}\cos\frac{2\pi x}{a}\cos\frac{\pi y}{b}$$

$$+ \frac{4(1+2\lambda^2)}{(1+8\lambda^2)}\cos\frac{\pi x}{a}\cos\frac{2\pi y}{b} + \frac{2+\lambda^2}{2(2+2\lambda^2)}\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}\Big\}$$

$$- \Big\{ \frac{1}{2\lambda^2}\cos\frac{\pi y}{b} + \frac{1}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} - \frac{1}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\cos\frac{\pi y}{b}\Big\}$$

$$+ a_0 + a_1 \cosh\frac{\sqrt{2}\,\lambda\pi x}{a}\cos\frac{\pi y}{b} \Big]$$

$$\left(\frac{\partial v}{\partial y}\right)_{y=\pm b} = \left(\frac{\sigma}{E}\right)_{y=\pm b} = \left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right) \lambda^2 \Big[\frac{f\pi^2}{32}\Big\{ \frac{3}{4} - \frac{4}{(1+2\lambda^2)}\cos\frac{\pi x}{a} - \frac{4}{(4+2\lambda^2)}\cos\frac{2\pi x}{a} + \frac{4(1+2\lambda^2)}{1+8\lambda^2}\cos\frac{\pi x}{a}$$

$$+ \frac{2+\lambda^2}{2(2+2\lambda^2)}\cos\frac{2\pi x}{a}\Big\} + \Big\{ \frac{1}{2\lambda^2} + \frac{1}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a} - \frac{1}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\Big\}$$

$$+ a_0 - a_1 \cosh\frac{\sqrt{2}\,\lambda\pi x}{a} \Big]$$

Average stress $= \left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right) \lambda^2\left[\frac{3}{128} f\pi^2 + \frac{1}{2\lambda^2} + a_0\right] = -\frac{\sigma}{E}$

The decrease in potential $\qquad \frac{\Delta\wp}{abEt} = \left[\left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right) \lambda^2\right]^2 a_0\left[\frac{3}{128} f\pi^2 + \frac{1}{2\lambda^2} + a_0\right]$

$$\boxed{\frac{\Delta\wp}{abEt} = +\frac{\sigma}{E}\left\{\frac{\sigma}{E} + \left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right) \lambda^2\left(\frac{3}{128} f\pi^2 + \frac{1}{2\lambda^2}\right)\right\}}$$

Decrease in Potential

$$\frac{v}{R} = \left(\frac{a}{R}\right)^3\left(\frac{f}{8}\right)\pi\lambda\left[\frac{f}{32}\left\{\frac{3}{8}\sin\frac{2\pi y}{b} + \frac{4}{(1+2\lambda^2)}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{2}{(2+\lambda^2)}\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}\right.\right.$$

$$\left. + \frac{2(1+2\lambda^2)}{(1+8\lambda^2)}\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b} + \frac{2+\lambda^2}{4(1+2\lambda^2)}\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\right\}$$

$$-\frac{1}{\pi^2}\left\{\frac{1}{2\lambda^2}\sin\frac{\pi y}{b} + \frac{1}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} - \frac{1}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}\right\}$$

$$\left. -\left(\frac{3}{128}f + \frac{1}{32\lambda^2}\right)\left(\frac{\pi y}{b}\right) + \frac{1}{\pi}a_1\cosh\frac{\sqrt{2}\,\lambda\pi x}{a}\sin\frac{\pi y}{b}\right] - \frac{\sigma}{E}\left(\frac{y}{R}\right)$$

$$\frac{\partial v}{\partial x} = \cdot\left(\frac{a}{R}\right)^2\left(\frac{f}{g}\right) \lambda\Bigg[\frac{f\pi^2}{32}\Bigg\{-\frac{4}{(1+2\lambda^2)} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} - \frac{4}{(2+\lambda^2)} \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b} - \frac{2(1+2\lambda^2)}{(1+8\lambda^2)} \sin\frac{\pi x}{a} \sin\frac{2\pi y}{b}$$

$$- \frac{2+\lambda^2}{2(1+2\lambda^2)} \sin\frac{2\pi x}{a} \sin\frac{2\pi y}{b}\Bigg\} + \Bigg\{\frac{[illegible](1+\lambda^2)}{(1+2\lambda^2)^2} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + \frac{1}{(1+2\lambda^2)} \sin\left(\frac{\pi x}{a}\right) \sin\frac{\pi y}{b}$$

$$+ \frac{1}{(1+2\lambda^2)} \left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi x}{a}\right) \sin\frac{\pi y}{b}\Bigg\}$$

$$+ \sqrt{2}\lambda a_1 \sinh\frac{\sqrt{2}\lambda\pi x}{a} \sin\frac{\pi y}{b}\Bigg]$$

$\left(\frac{\partial v}{\partial x}\right) = 0$ when $x = \pm a$,

$$\frac{1}{(1+2\lambda^2)}\pi = \sqrt{2}\lambda a_1 \sinh(\sqrt{2}\lambda\pi)$$

$$\boxed{a_1 = \frac{\pi}{(1+2\lambda^2)\sqrt{2}\lambda \sinh(\sqrt{2}\lambda\pi)}}$$

$$\frac{\partial v}{\partial y} = \left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right)\ \lambda^2\Big[\frac{f\pi^2}{32}\Big\{\frac{3}{4}\cos\frac{2\pi y}{b} + \frac{4}{(1+2\lambda^2)}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} + \frac{2}{(9+\lambda^2)}\cos\frac{2\pi x}{a}\cos\frac{\pi y}{b}$$

$$+\frac{4(1+2\lambda^2)}{(1+8\lambda^2)}\cos\frac{\pi x}{a}\cos\frac{2\pi y}{b} + \frac{2+\lambda^2}{4(1+2\lambda^2)}\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}\Big\}$$

$$-\Big\{\frac{1}{2\lambda^2}\cos\frac{\pi y}{b} + \frac{1}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} - \frac{1}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\cos\frac{\pi y}{b}\Big\}$$

$$-\left(\frac{3}{128}f\pi^2 + \frac{1}{2\lambda^2}\right) + a_1\cosh\frac{\sqrt{2}\lambda\pi x}{a}\cos\frac{\pi y}{b}\Big] - \frac{\sigma}{E}$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right)\ \lambda^2\Big[\frac{3}{64}f\pi^2 + \frac{1}{16}f\pi^2\cos\frac{\pi x}{a} + \frac{1}{64}f\pi^2\cos\frac{2\pi x}{a} - \frac{3}{64}f\pi^2\cos\frac{2\pi y}{b} - \frac{1}{16}f\pi^2\cos\frac{\pi x}{a}\cos\frac{2\pi y}{b}$$

$$-\frac{1}{64}f\pi^2\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}\Big]$$

$$\frac{\sigma_y}{E} = \left(\frac{a}{R}\right)^2\frac{f}{8}\ \lambda^2\Big[\left(\frac{3}{128}f\pi^2 - \frac{1}{2\lambda^2}\right) + \left(\frac{1}{16}f\pi^2\cos\frac{\pi x}{a} + \frac{1}{64}f\pi^2\cos\frac{2\pi x}{a}\right)$$

$$+\Big\{\frac{1}{8(1+2\lambda^2)}f\pi^2\cos\frac{\pi x}{a} + \frac{1}{16(2+\lambda^2)}f\pi^2\cos\frac{2\pi x}{a} - \frac{1}{2\lambda^2} - \frac{1}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a} + \frac{1}{(1+2\lambda^2)}\frac{\pi x}{a}\sin\frac{\pi x}{a} + a_1\cosh\frac{\sqrt{2}\lambda\pi x}{a}\Big\}\cos\frac{\pi y}{b}$$

$$+\Big\{-\frac{3}{128}f\pi^2 + \frac{1-4\lambda^2}{16(1+8\lambda^2)}f\pi^2\cos\frac{\pi x}{a} + \frac{1-\lambda^2}{64(1+2\lambda^2)}f\pi^2\cos\frac{2\pi x}{a}\Big\}\cos\frac{2\pi y}{b}\Big] - \frac{\sigma}{E}$$

$$\frac{1}{a}\int_0^a \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a} \cos\frac{\pi x}{a}\, dx = \frac{1}{\pi 8}\int_0^{2\pi} \theta \sin\theta\, d\theta = \frac{1}{8\pi}\Big[\sin\theta - \theta\cos\theta\Big]_0^{2\pi} = -\frac{1}{8\pi}2\pi = -\frac{1}{4}$$

$$\frac{1}{a}\int_0^a \cos\frac{\pi x}{a} \cosh(\sqrt{2}\lambda)\frac{\pi x}{a}\, dx = \frac{1}{2\pi}\int_0^a \left[\cosh\frac{\pi x}{a}(\sqrt{2}\lambda + i) + \cosh\frac{\pi x}{a}(\sqrt{2}\lambda - i)\right] d\left(\frac{\pi x}{a}\right)$$

$$= \frac{1}{2\pi}\left[\frac{1}{\sqrt{2}\lambda + i}\sinh\frac{\pi x}{a}(\sqrt{2}\lambda + i) + \frac{1}{\sqrt{2}\lambda - i}\sinh\frac{\pi x}{a}(\sqrt{2}\lambda - i)\right] = -\frac{1}{\pi}\frac{\sqrt{2}\lambda}{2\lambda^2+1}\sinh\sqrt{2}\lambda\pi$$

$$\frac{1}{a}\int_0^a \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a} \cos\frac{2\pi x}{a}\, dx = \frac{1}{2\pi}\int_0^a \left(\frac{\pi x}{a}\right)\left[\sin\frac{3\pi x}{a} - \sin\frac{\pi x}{a}\right] d\left(\frac{\pi}{a}\right)$$

$$= \frac{1}{2\pi}\left[\frac{1}{9}\left(\sin\theta - \theta\cos\theta\right)_0^{3\pi} - \left(\sin\theta - \theta\cos\theta\right)_0^{\pi}\right] = \frac{1}{2}\left(\frac{1}{3} - 1\right) = -\frac{1}{3}$$

$$\frac{1}{a}\int_0^a \cos\frac{2\pi x}{a} \cosh(\sqrt{2}\lambda)\frac{\pi x}{a}\, dx = \frac{1}{\pi}\,\frac{\sqrt{2}\lambda \sinh\sqrt{2}\lambda\pi}{2\lambda^2+4}$$

$$\frac{1}{a}\int_0^a \frac{\pi x}{a}\sin\frac{\pi x}{a}\, dx = 1$$

$$\frac{1}{a}\int_0^a \cosh\frac{\sqrt{2}\lambda\pi x}{a}\, dx = \frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}$$

$$\frac{1}{a}\int_0^a \left(\frac{\pi x}{a}\right)^2 \sin^2\frac{\pi x}{a}\, dx = \frac{1}{2\pi}\int_0^a \left\{\left(\frac{\pi x}{a}\right)^2 - \left(\frac{\pi x}{a}\right)^2\cos\frac{2\pi x}{a}\right\} d\left(\frac{\pi x}{a}\right) = \frac{1}{2\pi}\left[\frac{\pi^3}{3} - \frac{1}{8}\left\{2\theta\cos\theta + (\theta^2-2)\sin\theta\right\}_0^{2\pi}\right]$$

$$= \frac{\pi^2}{6} - \frac{1}{16\pi}(4\pi) = \frac{\pi^2}{6} - \frac{1}{4}$$

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$$\frac{1}{a}\int_0^a \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a} \cosh(\sqrt{2}\lambda)\frac{\pi x}{a}\,dx = \frac{1}{2\pi i}\int_0^a \left[\left(\frac{\pi x}{a}\right)\sinh(\sqrt{2}\lambda+i) - \left(\frac{\pi x}{a}\right)\sinh(\sqrt{2}\lambda-i)\right] d\left(\frac{\pi x}{a}\right)$$

$$= \frac{1}{2\pi i}\left[\frac{1}{(\sqrt{2}\lambda+i)^2}\left[\left(\frac{\pi x}{a}\right)(\sqrt{2}\lambda+i)\cosh\frac{\pi x}{a}(\sqrt{2}\lambda+i) - \sinh\frac{\pi x}{a}(\sqrt{2}\lambda+i)\right]\right.$$

$$\left. - \frac{1}{(\sqrt{2}\lambda-i)^2}\left[\left(\frac{\pi x}{a}\right)(\sqrt{2}\lambda-i)\cosh\frac{\pi x}{a}(\sqrt{2}\lambda-i) - \sinh\frac{\pi x}{a}(\sqrt{2}\lambda-i)\right]\right]_0^a$$

$$= \frac{1}{2\pi i}\left[\frac{\pi x}{a}\left\{\frac{\cosh\frac{\pi x}{a}(\sqrt{2}\lambda+i)}{\sqrt{2}\lambda+i} - \frac{\cosh\frac{\pi x}{a}(\sqrt{2}\lambda-i)}{\sqrt{2}\lambda-i}\right\} - \left\{\frac{\sinh\frac{\pi x}{a}(\sqrt{2}\lambda+i)}{(2\lambda^2-1)+2\sqrt{2}\lambda i} - \frac{\sinh\frac{\pi x}{a}(\sqrt{2}\lambda-i)}{(2\lambda^2-1)-2\sqrt{2}\lambda i}\right\}\right]$$

$$= \frac{\cosh\sqrt{2}\lambda\pi}{2\lambda^2+1} - \frac{2\sqrt{2}\lambda\sinh\sqrt{2}\lambda\pi}{\pi\left[(2\lambda^2-1)^2+8\lambda^2\right]} = \frac{\cosh\sqrt{2}\lambda\pi}{2\lambda^2+1} - \frac{2\sqrt{2}\lambda\sinh\sqrt{2}\lambda\pi}{\pi(2\lambda^2+1)^2}$$

$$\frac{1}{a}\int_0^a \cosh^2(\sqrt{2}\lambda)\frac{\pi x}{a}\,dx = \frac{1}{2a}\int_0^a\left[1+\cosh(2\sqrt{2}\lambda)\frac{\pi x}{a}\right]dx = \frac{1}{2} + \frac{1}{4\sqrt{2}\lambda\pi}\sinh 2\sqrt{2}\lambda\pi$$

$$\frac{1}{2E^2ab}\int_0^a\int_0^b \sigma_y^2\,dx\,dy = \frac{1}{2}\left(\frac{\sigma}{E}\right)^2 - \left(\frac{a}{R}\right)^2\frac{f}{8}\;\lambda^2\frac{\sigma}{E}\Big[\Big(\frac{3}{128}f\pi^2 - \frac{1}{2\lambda^2}\Big)\Big]$$

$$+\left(\frac{a}{R}\right)^4\left(\frac{f\pi\,\lambda^2}{8}\right)^2\Big[\frac{1}{2}\Big(\frac{3}{128}f\pi^2-\frac{1}{2\lambda^2}\Big)^2+\frac{1}{4}\Big\{\Big(\frac{f\pi}{16}\Big)^2+\Big(\frac{f\pi}{64}\Big)^2\Big\}$$

$$+\frac{1}{8}\Big\{2\Big(\frac{3}{128}f\pi\Big)^2+\Big(\frac{1-4\lambda^2}{16(1+8\lambda^2)}f\pi\Big)^2+\Big(\frac{1-\lambda^2}{64(1+2\lambda^2)}f\pi\Big)^2+\Big(\frac{f\pi^2}{8(1+2\lambda^2)}-\frac{1}{(1+2\lambda^2)^2}\Big)^2+\Big(\frac{f\pi^2}{16(2+\lambda^2)}\Big)^2+2\Big(\frac{1}{2\lambda^2}\Big)^2\Big\}$$

$$+\frac{1}{4a}\Big\{2\int_0^a\Big(\frac{f\pi^2}{8(1+2\lambda^2)}-\frac{1}{(1+2\lambda^2)^2}\Big)\Big(\frac{1}{1+2\lambda^2}\frac{\pi x}{a}\sin\frac{\pi x}{a}+a_1\cosh\frac{\sqrt{2}\,\lambda\pi x}{a}\Big)\cos\frac{\pi x}{a}\,dx$$

$$+2\int_0^a\frac{f\pi^2}{16(2+\lambda^2)}\Big(\frac{1}{1+2\lambda^2}\frac{\pi x}{a}\sin\frac{\pi x}{a}+a_1\cosh\frac{\sqrt{2}\,\lambda\pi x}{a}\Big)\cos\frac{2\pi x}{a}\,dx$$

$$-\frac{1}{2\lambda^2}\int_0^a 2\Big(\frac{1}{1+2\lambda^2}\frac{\pi x}{a}\sin\frac{\pi x}{a}+a_1\cosh\frac{\sqrt{2}\,\lambda\pi x}{a}\Big)\,dx$$

$$+\int_0^a\Big(\frac{1}{1+2\lambda^2}\frac{\pi x}{a}\sin\frac{\pi x}{a}+a_1\cosh\frac{\sqrt{2}\,\lambda\pi x}{a}\Big)^2dx\Big\}\Big]$$

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$$\frac{1}{2E^2ab}\int_0^a\int_0^b \sigma_y^2\,dx\,dy = \frac{1}{2}\left(\frac{\sigma}{E}\right)^2 + \left(\frac{a}{R}\right)^2\frac{f^2}{8}\;\lambda^2\frac{\sigma}{E}\left(\frac{1}{9\lambda^2}-\frac{3}{128}f\pi^2\right)$$

$$+\left(\frac{a}{R}\right)^4\left(\frac{1}{8}\lambda^2\right)^2\Big[\frac{1}{8}\left(\frac{3}{64}f\pi^2-\frac{1}{\lambda^2}\right)^2+\frac{17}{65536}(f\pi^2)^2+\frac{(1-4\lambda^2)^2}{2048(1+8\lambda^2)^2}(f\pi^2)^2+\frac{1}{32768}\frac{(1-\lambda^2)^2}{(1+2\lambda^2)^2}(f\pi^2)^2+\frac{1}{512}\frac{1}{(1+2\lambda^2)^2}(f\pi^2)^2$$

$$-\frac{1}{32}\frac{1}{(1+2\lambda^2)^3}(f\pi^2)+\frac{1}{8(1+2\lambda^2)^4}+\frac{1}{2048}\frac{1}{(2+\lambda^2)^2}(f\pi^2)^2+\frac{1}{16}\frac{1}{\lambda^4}+\frac{1}{2}\left\{\frac{f\pi^2}{8(1+2\lambda^2)}-\frac{1}{(1+2\lambda^2)^2}\right\}\left\{-\frac{1}{4}\frac{1}{(1+2\lambda^2)}\right.$$

$$\left.-\frac{1}{(1+2\lambda^2)^2}\right\}+\frac{1}{2}\frac{f\pi^2}{16(2+\lambda^2)}\left\{-\frac{1}{3(1+2\lambda^2)}+\frac{1}{2}\frac{1}{(1+2\lambda^2)(2+\lambda^2)}\right\}-\frac{1}{4\lambda^2}\frac{1}{(1+2\lambda^2)}-\frac{1}{4\lambda^2}\frac{1}{2\lambda^2(1+2\lambda^2)}$$

$$+\frac{1}{4}\frac{1}{(1+2\lambda^2)^2}\left(\frac{\pi^2}{6}-\frac{1}{4}\right)+\frac{1}{2}\frac{1}{(1+2\lambda^2)^3}\frac{\pi}{\sqrt{2}\lambda}\coth\sqrt{2}\lambda\pi-\frac{1}{2}\frac{2}{(1+2\lambda^2)^4}+\frac{1}{4}\pi^2\frac{1}{(1+2\lambda^2)^2\lambda^2}\frac{\frac{1}{2}+\frac{1}{4\sqrt{2}\lambda\pi}\sinh 2\sqrt{2}\lambda\pi}{(\cosh 2\sqrt{2}\lambda\pi-1)}\Big]$$

$$\frac{\sigma_x}{E}=\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2=\left(\frac{a}{R}\right)^2\left(\frac{f\pi}{8}\right)\Big[-\left(\frac{x}{a}\right)\sin\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)+\left(\frac{f\pi}{8}\right)\frac{1}{8}\left(1-\cos\frac{2\pi x}{a}\right)\left(3+4\cos\frac{\pi y}{b}+\cos\frac{2\pi y}{b}\right)\Big]$$

$$=\left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right)\Big[-\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}-\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}+\frac{f\pi^2}{64}\Big\{3+4\cos\frac{\pi y}{b}+\cos\frac{2\pi y}{b}-3\cos\frac{2\pi x}{a}-4\cos\frac{2\pi x}{a}\cos\frac{\pi y}{b}$$

$$-\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}\Big\}\Big]$$

$$=\left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right)\Big[\left\{-\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}+\frac{3}{64}f\pi^2-\frac{3}{64}f\pi^2\cos\frac{2\pi x}{a}\right\}$$

$$+\left\{-\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}+\frac{f\pi^2}{16}-\frac{f\pi^2}{16}\cos\frac{2\pi x}{a}\right\}\cos\frac{\pi y}{b}$$

$$+\left\{\frac{f\pi^2}{64}-\frac{f\pi^2}{64}\cos\frac{2\pi x}{a}\right\}\cos\frac{2\pi y}{b}\Big]$$

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$$\frac{1}{\gamma E^2 ab}\int_0^a\int_0^b \sigma_x^2\,dy\,dx = \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\left[\frac{3}{4}\left(\frac{3}{64}\right)^2(f\pi)^2 + \frac{1}{a}\frac{3}{64}(f\pi)\int_0^a\left(\cos\frac{2\pi x}{a}-1\right)\left(\frac{x}{a}\right)\sin\frac{\pi x}{a}\,dx + \frac{3}{4a}\int_0^a\left(\frac{\pi x}{a}\right)^2\sin^2\frac{\pi x}{a}\,dx\right.$$

$$\left. + \frac{3}{8}\left(\frac{f\pi^2}{16}\right)^2 - \frac{1}{2a}\left(\frac{f\pi^2}{16}\right)\int_0^a\left(1-\cos\frac{2\pi x}{a}\right)\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\,dx + \frac{1}{8a}\left(\frac{f\pi^2}{64}\right)^2 3\right].$$

$$= \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\left[\cancel{\frac{25}{16384}}(f\pi)^2 + \frac{3}{64}(f\pi)\left\{-\frac{1}{3}-1\right\} + \frac{3}{4}\left\{\frac{\pi^2}{6}-\frac{1}{4}\right\} - \frac{f\pi^2}{32}\left\{1+\frac{1}{3}\right\} + \cancel{\frac{3}{16384}(f\pi)^2}\right]$$

$$= \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\left[\frac{105}{32768}(f\pi^2)^2 - \frac{5}{48}(f\pi^2) + \frac{3}{2}\left(\frac{\pi^2}{12}-\frac{1}{8}\right)\right]$$

$$\frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \left(\frac{a}{R}\right)^2\left(\frac{f\pi}{8}\lambda\right)\left[-\frac{1}{2}\left(\frac{x}{a}\right)\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{b} + \left(\frac{f\pi}{16}\right)\left(\sin\frac{\pi x}{a}+\frac{1}{2}\sin\frac{2\pi x}{a}\right)\left(\sin\frac{\pi y}{b}+\frac{1}{2}\sin\frac{2\pi y}{b}\right)\right]$$

$$\frac{\tau_{xy}}{E} = \left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right)\lambda\left[\cdot\frac{1}{2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi y}{b} - \frac{1}{2}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \left(\frac{f\pi^2}{16}\right)\left\{\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{1}{2}\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b}\right.\right.$$

$$\left. + \frac{1}{2}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b} + \frac{1}{4}\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\right\}$$

$$+ \frac{1+\lambda^2}{(1+2\lambda^2)^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{1}{2(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)\sin\frac{\pi y}{b} + \frac{1}{2}\sqrt{2}\lambda a_1\sinh\frac{\sqrt{2}\lambda\pi x}{a}\sin\frac{\pi y}{b}$$

$$+ \frac{(f\pi^2)}{64}\left\{-\frac{4}{(1+2\lambda^2)}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} - \frac{4}{(2+\lambda^2)}\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b} - \frac{2(1+2\lambda^2)}{(1+8\lambda^2)}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b}\right.$$

$$\left.\left. - \frac{2+\lambda^2}{2(1+2\lambda^2)}\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\right\}\right]$$

$$\frac{\tau_{xy}}{E} = \left(\frac{a}{R}\right)^2\left(\frac{f\lambda}{8}\right)\left[\left\{\left(-\frac{1}{2}\left(\frac{\pi x}{a}\right) - \frac{1}{2}\left(\frac{\pi x}{a}\right)\right)\cos\frac{\pi x}{a} + \frac{f\pi^2}{16}\sin\frac{\pi x}{a} + \frac{f\pi^2}{32}\sin\frac{2\pi x}{a} + \frac{1+\lambda^2}{(1+2\lambda^2)^2}\sin\frac{\pi x}{a} + \frac{1}{2(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)\right.\right.$$

$$+\frac{1}{2}\sqrt{2}\lambda a_1 \sinh\frac{\sqrt{2}\lambda\pi x}{a} - \frac{f\pi^2}{16}\frac{1}{(1+2\lambda^2)}\sin\frac{\pi x}{a} - \frac{f\pi^2}{16}\frac{1}{(2+\lambda^2)}\sin\frac{2\pi x}{a}\Big\}\sin\frac{\pi y}{b}$$

$$+\left\{\frac{f\pi^2}{32}\sin\frac{\pi x}{a} + \frac{f\pi^2}{64}\sin\frac{2\pi x}{a} - \frac{f\pi^2}{32}\frac{(1+2\lambda^2)}{(1+8\lambda^2)}\sin\frac{\pi x}{a} - \frac{f\pi^2}{128}\frac{2+\lambda^2}{1+2\lambda^2}\sin\frac{2\pi x}{a}\right\}\sin\frac{2\pi y}{b}\Big]$$

$$= \left(\frac{a}{R}\right)^2\left(\frac{f\lambda}{8}\right)\left[\left\{-\frac{1}{2}\left(\frac{\pi x}{a}\right) - \frac{\lambda^2}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a} + \frac{f\pi^2}{8}\frac{\lambda^2}{(1+2\lambda^2)}\sin\frac{\pi x}{a} + \frac{f\pi^2}{32}\frac{\lambda^2}{(2+\lambda^2)}\sin\frac{2\pi x}{a} + \frac{1+\lambda^2}{(1+2\lambda^2)^2}\sin\frac{\pi x}{a}\right.\right.$$

$$+\frac{1}{2}\sqrt{2}\lambda a_1\sinh\frac{\sqrt{2}\lambda\pi x}{a}\Big\}\sin\frac{\pi y}{b}$$

$$+\left\{\frac{f\pi^2}{16}\frac{3\lambda^2}{(1+8\lambda^2)}\sin\frac{\pi x}{a} + \frac{f\pi^2}{128}\frac{3\lambda^2}{(1+2\lambda^2)}\sin\frac{2\pi x}{a}\right\}\sin\frac{2\pi y}{b}\Big]$$

$$= \left(\frac{a}{R}\right)^2\left(\frac{f\lambda}{8}\right)\left[\left\{\left[\frac{f\pi^2}{8}\frac{\lambda^2}{(1+2\lambda^2)} + \frac{1+\lambda^2}{(1+2\lambda^2)^2}\right]\sin\frac{\pi x}{a} + \frac{f\pi^2}{32}\frac{\lambda^2}{(2+\lambda^2)}\sin\frac{2\pi x}{a} + \frac{1}{2}\sqrt{2}\lambda a_1\sinh\frac{\sqrt{2}\lambda\pi x}{a}\right.\right.$$

$$-\frac{1}{2}\left(\frac{\pi x}{a}\right) - \frac{\lambda^2}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\Big\}\sin\frac{\pi y}{b}$$

$$+\left\{\frac{f\pi^2}{16}\frac{3\lambda^2}{(1+8\lambda^2)}\sin\frac{\pi x}{a} + \frac{f\pi^2}{128}\frac{3\lambda^2}{(1+2\lambda^2)}\sin\frac{2\pi x}{a}\right\}\sin\frac{2\pi y}{b}\Big]$$

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$$\frac{1}{E^2 ab}\int_0^a\int_0^b \tau_{xy}^2\,dx\,dy = \left(\frac{a}{R}\right)^2\left(\frac{f\lambda}{8}\right)^2\Big[\frac{1}{4}\Big\{\frac{f\pi^2}{8}\frac{\lambda^2}{(1+2\lambda^2)}+\frac{1+\lambda^2}{(1+2\lambda^2)^2}\Big\}^2+\frac{1}{4}\left(\frac{f\pi^2}{32}\right)^2\left(\frac{\lambda^2}{2+\lambda^2}\right)^2$$

$$+\frac{1}{4}\left(\frac{f\pi^2}{16}\right)^2\left(\frac{3\lambda^2}{1+8\lambda^2}\right)^2+\frac{1}{4}\left(\frac{f\pi^2}{128}\right)^2\left(\frac{3\lambda^2}{1+2\lambda^2}\right)^2+\cancel{\frac{1}{4}\left(\frac{f\pi^2}{16}\right)^2\left(\frac{3\lambda^2}{1+8\lambda^2}\right)^2}+\cancel{\frac{1}{4}\left(\frac{f\pi^2}{128}\right)^2\left(\frac{3\lambda^2}{1+2\lambda^2}\right)^2}$$

$$+\Big[\frac{f\pi^2}{8}\frac{\lambda^2}{(1+2\lambda^2)}+\frac{1+\lambda^2}{(1+2\lambda^2)^2}\Big]\frac{1}{a}\int_0^a \sin\frac{\pi x}{a}\Big\{\frac{1}{2}\sqrt{2}\lambda a_1\sinh\frac{\sqrt{2}\lambda\pi x}{a}-\frac{1}{2}\frac{\pi x}{a}-\frac{\lambda^2}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\Big\}dx$$

$$+\frac{f\pi^2}{32}\frac{\lambda^2}{(2+\lambda^2)}\frac{1}{a}\int_0^a \sin\frac{2\pi x}{a}\Big\{\frac{1}{2}\sqrt{2}\lambda a_1\sinh\frac{\sqrt{2}\lambda\pi x}{a}-\frac{1}{2}\frac{\pi x}{a}-\frac{\lambda^2}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\Big\}dx$$

$$+\frac{1}{2}\frac{1}{a}\int_0^a\Big\{\frac{1}{2}\sqrt{2}\lambda a_1\sinh\frac{\sqrt{2}\lambda\pi x}{a}-\frac{1}{2}\frac{\pi x}{a}-\frac{\lambda^2}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\Big\}^2dx$$

$$\frac{1}{a}\int_0^a \sin\frac{\pi x}{a}\sinh(\sqrt{2}\lambda)\left(\frac{\pi x}{a}\right)dx = \frac{1}{2\pi i}\int_0^\pi\Big[\cosh\theta(\sqrt{2}\lambda+i)-\cosh(\sqrt{2}\lambda-i)\theta\Big]d\theta$$

$$=\frac{1}{2\pi i}\Big[\frac{1}{\sqrt{2}\lambda+i}\sinh\theta(\sqrt{2}\lambda+i)-\frac{1}{\sqrt{2}\lambda-i}\sinh\theta(\sqrt{2}\lambda-i)\Big]_0^\pi=+\frac{1}{\pi}\frac{1}{(1+2\lambda^2)}\sinh(\sqrt{2}\lambda\pi)$$

$$\frac{1}{a}\int_0^a\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}dx=\frac{1}{\pi}\Big[\sin\theta-\theta\cos\theta\Big]_0^\pi=1$$

$$\frac{1}{a}\int_0^a\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\sin\frac{\pi x}{a}dx=\frac{1}{8\pi}\Big[\sin\theta-\theta\cos\theta\Big]_0^{2\pi}=-\frac{1}{4}$$

$$\frac{\Delta \wp}{abtE} = -\left(\frac{a}{R}\right)^2 \frac{(f\lambda)^2}{64}\left(\frac{3}{2}\pi^2 \frac{\sigma}{E}\right)$$

$$\frac{t^2}{12ab}\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w_0}{\partial x^2}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{2}\right)^2\frac{3}{4}$$

$$\frac{t^2}{12ab}\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial y^2}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2\lambda^2}{2}\right)^2\frac{3}{4}$$

$$\frac{t^2}{12ab}\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)\left(\frac{f\pi^2\lambda}{2}\right)^2\frac{1}{4}$$

The part of extensional energy which is independent of a's b's

$$= \left(\frac{a}{R}\right)^4 \frac{(f\lambda^2)^2}{64}\Bigg[(f\pi^2)^2 \Big\{ \frac{1}{32} + \frac{1}{16(1+\lambda^2)^4} + \frac{1}{4(4+\lambda^2)^4} + \frac{1}{64(1+4\lambda^2)^4} + \frac{1}{512} + \frac{1}{32\lambda^4} + \frac{\lambda^4}{16(1+\lambda^2)^4} + \frac{1}{512\lambda^4} +$$

$$+ \frac{\lambda^4}{64(4+\lambda^2)^4} + \frac{\lambda^4}{4(1+4\lambda^2)^4} + \frac{2\lambda^2}{16(1+\lambda^2)^4} + \frac{2\lambda^2}{16(4+\lambda^2)^4} + \frac{2\lambda^2}{16(1+4\lambda^2)^4} \Big\}$$

$$- (f\pi^2) \Big\{ \frac{1}{4(1+\lambda^2)^4} + \frac{1}{4\lambda^4} + \frac{\lambda^4}{4(1+\lambda^2)^4} + \frac{2\lambda^2}{4(1+\lambda^2)^4} \Big\}$$

$$+ \Big\{ \frac{1}{4(1+\lambda^2)^4} + \frac{1}{2\lambda^4} + \frac{\lambda^4}{4(1+\lambda^2)^4} + \frac{2\lambda^2}{4(1+\lambda^2)^4} \Big\} \Bigg]$$

$$= \left(\frac{a}{R}\right)^4 \frac{(f\lambda^2)^2}{64}\Bigg[(f\pi^2)^2 \Big\{ \frac{17}{512} + \frac{17}{512\lambda^4} + \frac{1}{16(1+\lambda^2)^2} + \frac{1}{64(4+\lambda^2)^2} + \frac{1}{64(1+4\lambda^2)^2} \Big\}$$

$$- (f\pi^2) \Big\{ \frac{1}{4\lambda^4} + \frac{1}{4(1+\lambda^2)^2} \Big\} + \Big\{ \frac{1}{2\lambda^4} + \frac{1}{4(1+\lambda^2)^2} \Big\} \Bigg]$$

The coefficient of $\left\{ \underline{\underline{a_0}}\ \frac{(f\lambda^2)^2}{64}\ \underline{\underline{\frac{\sinh 2\pi}{2\pi}}} \right\} \left(\frac{a}{R}\right)^4$ in $(A_1 + A_2 + 2A_3)$

$(f\pi^2)$

$$-\frac{1}{4} - \frac{1}{4(1+\lambda^2)^2} - \frac{1}{20(1+4\lambda^2)^2} + \frac{1}{4} + \frac{\lambda^4}{4(1+\lambda^2)^2} + \frac{\lambda^4}{5(1+4\lambda^2)^2}$$

$$- \frac{2\lambda^2}{4(1+\lambda^2)^2} - \frac{2\lambda^2}{5(1+4\lambda^2)^2}$$

$$= -\frac{1}{4(1+\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2} - \frac{1}{20(1+4\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)}$$

$$+ \frac{1}{2(1+\lambda^2)^2} - 1 - \frac{\lambda^4}{2(1+\lambda^2)^2} + \frac{2\lambda^2}{2(1+\lambda^2)^2}$$

$$= \Big(-1 + \frac{1}{2(1+\lambda^2)^2} + \frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2}$$

$\left\{ b_2\ \frac{(f\lambda^2)^2}{64}\ \frac{\sinh 2\pi}{2\pi} \right\} \left(\frac{a}{R}\right)^4$

$$(f\pi^2)\Big\{ + \frac{1}{4} - \frac{3}{100(1+4\lambda^2)^2} + \frac{1}{4} + \frac{\lambda^4}{2(1+\lambda^2)^2} + \frac{2\lambda^4}{5(1+4\lambda^2)^2} + \frac{3\lambda^4}{25(1+4\lambda^2)^2}$$

$$- \frac{2\lambda^2}{5(1+4\lambda^2)^2} + \frac{4\lambda^2}{25(1+4\lambda^2)^2} \Big\}$$

$$= (f\pi^2)\left\{ \frac{1}{2} + \frac{\lambda^4}{2(1+\lambda^2)^2} + \frac{2}{5}\,\frac{\lambda^2(\lambda^2-1)}{(1+4\lambda^2)^2} + \frac{1}{100}\,\frac{(2\lambda^2+3)(6\lambda^2-1)}{(1+4\lambda^2)^2} \right\}$$

$$\left\{ -1 - \frac{\lambda^4}{(1+\lambda^2)^2} \right\} =$$

$$\underbrace{\left\{ b_2 \frac{(4+\lambda^2)^2}{64} \cosh 2\pi \right\} \left(\frac{a}{R}\right)^4}$$

$$(4+\pi^2)\left\{ -\frac{1}{4} - \frac{1}{4(1+\lambda^2)^2} - \frac{1}{20(1+4\lambda^2)^2} + \frac{1}{4} + \frac{\lambda^4}{4(1+\lambda^2)} + \frac{\lambda^4}{5(1+4\lambda^2)^2} \right.$$

$$\left. - \frac{2\lambda^2}{4(1+\lambda^2)^2} - \frac{2\lambda^2}{5(1+4\lambda^2)^2} \right\}$$

$$= (4+\pi^2)\left\{ -\frac{1}{4(1+\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2} - \frac{1}{20(1+4\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)} \right\}$$

$$\left\{ \frac{1}{2(1+\lambda^2)^2} - 1 - \frac{\lambda^4}{2(1+\lambda^2)} + \frac{2\lambda^2}{2(1+\lambda^2)^2} \right\}$$

$$= \frac{1}{2(1+\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2} - 1$$

$$\underbrace{\left\{ \boxed{a_4 \frac{(4+\lambda^2)^2}{64} \frac{\sinh 4\pi}{4\pi}} \right\} \left(\frac{a}{R}\right)^4}$$

$$4\pi^2)\left\{ -\frac{1}{16} - \frac{4}{5}\frac{1}{(4+\lambda^2)^2} + \frac{1}{16} + \frac{\lambda^4}{5(4+\lambda^2)^2} - \frac{2\lambda^2}{5(4+\lambda^2)^2} \right\}$$

$$= 4\pi^2)\left\{ -\frac{1}{5}\frac{1}{(4+\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2} \right\}.$$

$$\underbrace{\left\{ b_4 \frac{(4+\lambda^2)^2}{64} \frac{\sinh 4\pi}{4\pi} \right\} \left(\frac{a}{R}\right)^4}$$

$$(4\pi^2)\left\{ +\frac{1}{16} + \frac{12}{25}\frac{1}{(4+\lambda^2)^2} + \frac{1}{16} + \frac{2}{5}\frac{\lambda^4}{(4+\lambda^2)^2} - \frac{3}{25}\frac{\lambda^4}{(4+\lambda^2)^2} - \frac{2\lambda^2}{5(4+\lambda^2)} \right.$$

$$\left. + \frac{16\lambda^2}{25(4+\lambda^2)^2} \right\}$$

$$= (4\pi^2)\left\{ \frac{1}{8} + \frac{(2+3\lambda^2)(6-\lambda^2)}{25(4+\lambda^2)^2} - \frac{2}{5}\frac{\lambda^2(1-\lambda^2)}{(4+\lambda^2)^2} \right\}$$

$$\frac{\left\{b_4 \frac{4\lambda^2}{64}\cosh 4\pi\right\}\left(\frac{a}{R}\right)^4}{H\pi^2}\left\{-\frac{1}{16}-\frac{4}{5}\frac{1}{(4+\lambda^2)^2}+\frac{1}{16}+\frac{1}{5}\frac{\lambda^4}{(4+\lambda^2)^2}-\frac{2\lambda^2}{5(4+\lambda^2)^2}\right\}$$

$$= H\pi^2\left\{-\frac{1}{5}\frac{1}{(4+\lambda^2)}-\frac{1}{5}\frac{\lambda^2(1-\lambda^2)}{(4+\lambda^2)^2}\right\}$$

Terms in $A_1+A_2+2A_3$ involving quadratics of a's & b's. $\div \frac{4\lambda^2}{64}\left(\frac{a}{R}\right)^4$

$$\frac{1}{4}\left\{(1+\lambda^2)^2\frac{\sinh 4\pi}{4\pi}+(1-\lambda^2)^2\right\}a_2^2$$

$$+\frac{1}{4}\left\{(3\lambda^4+2\lambda^2-1)\frac{\sinh 4\pi}{4\pi}+(\lambda^2+1)^2\cosh 4\pi-4(1-\lambda^2)\lambda^2\right\}a_2 b_2$$

$$+\frac{1}{4}\left\{\left[4\pi^2(\lambda^2+1)^2+\frac{1}{2}(5\lambda^4+2\lambda^2+1)\right]\frac{\sinh 4\pi}{4\pi}+\frac{1}{2}(3\lambda^4+2\lambda^2-1)\cosh 4\pi\right.$$
$$\left.-\frac{4\pi^2}{3}(1-\lambda^2)^2+2(2\lambda^4-1)\right\}b_2^2$$

$$\frac{1}{4}\left\{(1+\lambda^2)^2\frac{\sinh 8\pi}{8\pi}+(1-\lambda^2)^2\right\}a_4^2$$

$$+\frac{1}{4}\left\{(3\lambda^4+2\lambda^2-1)\frac{\sinh 8\pi}{8\pi}+(\lambda^2+1)^2\cosh 8\pi+4\lambda^2(\lambda^2-1)\right\}a_4 b_4$$

$$+\frac{1}{4}\left\{\left[16\pi^2(\lambda^2+1)^2+\frac{1}{2}(5\lambda^4+2\lambda^2+1)\right]\frac{\sinh 8\pi}{8\pi}+\frac{1}{2}(3\lambda^4+2\lambda^2-1)\cosh 8\pi\right.$$
$$\left.-\frac{16\pi^2}{3}(1-\lambda^2)^2+2(2\lambda^4-1)\right\}b_4^2$$

①	②	③	④	⑤	⑥	⑦	⑧	⑨	
λ	$\sqrt{2}\lambda\pi$	② ÷ 2.3025851	$e^{②}$	$e^{-②}$	$\cosh\sqrt{2}\lambda\pi$	$\sinh\sqrt{2}\lambda\pi$	$\sinh\sqrt{2}\lambda\pi / \sqrt{2}\lambda\pi$	⑦/⑧	⑧^2
2.0	8.885768	3.8590400	7228.363	0	3614.182	3614.182	406.7383	8.885768	
1.9	8.441480	3.6660882	4635.411	0	2317.706	2317.706	274.5616	8.441480	
1.8	7.997191	3.4731359	2972.595	0	1486.298	1486.298	185.8525	7.997191	
1.7	7.552903	3.2801841	1906.268	0.001	953.1345	953.1335	126.1943	7.552912	
1.6	7.108614	3.0872318	1222.452	0.001	611.2265	611.2255	85.98378	7.108626	
1.5	6.664326	2.8942800	783.9349	0.0013	391.9681	391.9668	58.81468	6.664460	
1.4	6.220038	2.7013282	502.7223	0.0020	251.3622	251.3602	40.41136	6.220088	
1.3	5.775749	2.5083759	322.3858	0.0031	161.1945	161.1914	27.90831	5.775860	
1.2	5.331461	2.3154241	206.7398	0.0049	103.3724	103.3675	19.38821	5.331714	
1.1	4.887172	2.1224718	132.5781	0.0076	66.29285	66.28525	13.56311	4.887732	
1.0	4.442884	1.9295200	85.01977	0.01176	42.51577	42.50401	9.566761	4.444113	
0.9	3.998596	1.7365682	54.52155	0.01834	27.26995	27.25161	6.815295	4.001287	
0.8	3.554307	1.5436159	34.96358	0.02860	17.49609	17.46749	4.914457	3.560127	
0.7	3.110019	1.3506641	22.42147	0.04460	11.23304	11.18844	3.597547	3.122416	
0.6	2.665730	1.1577118	14.37874	0.06955	7.223995	7.154445	2.683860	2.691644	
0.5	2.221442	0.9647600	9.220617	0.108453	4.664536	4.556083	2.050957	2.274322	
0.4	1.777154	0.7718082	5.913004	0.169119	3.041062	2.871943	1.616035	1.881805	

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λ	(10) λ^4	(11) λ^2	(12) $95\lambda^4/65536$	(13) $1+2\lambda^2$	(14) $1+8\lambda^2$	(15) $130+\lambda^2$	(16) $(1+2\lambda^2)^2$	(17) $(1+2\lambda^2)/2048(1+\lambda^2)$	(18) $(130+\lambda^2)/65536(1+\lambda^2)$	(19) $3\lambda^2/65536(1+\lambda^2)^2$
2.0	16.0000	4.00	0.02319336	9.00	33.00	134.00	81.00	0.00001331676	0.0002271864	0.0000361690
1.9	13.0321	3.61	0.01889114	8.22	29.88	133.61	67.5684	0.00001343264	0.0002480203	0.0000391314
1.8	10.4976	3.24	0.01521716	7.48	26.92	133.24	55.9504	0.00001356740	0.0002718023	0.0000424134
1.7	8.3521	2.89	0.01210708	6.78	24.12	132.89	45.9684	0.00001372532	0.0002990768	0.0000460468
1.6	6.5536	2.56	0.0095000	6.12	21.48	132.56	37.4544	0.00001391192	0.0003305074	0.0000500609
1.5	5.0625	2.25	0.00733852	5.50	19.00	132.25	30.2500	0.00001413446	0.0003669045	0.0000544777
1.4	3.8416	1.96	0.00556873	4.92	16.68	131.96	24.0640	0.00001440254	0.0004092581	0.0000596554
1.3	2.8561	1.69	0.00414016	4.38	14.52	131.69	19.1844	0.00001472915	0.0004587740	0.0000645208
1.2	2.0736	1.44	0.00300586	3.88	12.52	131.44	15.0544	0.00001513204	0.0005169111	0.0000700584
1.1	1.4641	1.21	0.00212234	3.42	10.68	131.21	11.6964	0.00001563597	0.0005854110	0.0000757675
1.0	1.0000	1.00	0.00144959	3.00	9.00	131.00	9.0000	0.00001627604	0.0006663005	0.0000813802
0.9	0.6561	0.81	0.00095107	2.62	7.48	130.81	6.8644	0.00001710290	0.0007618329	0.0000864259
0.8	0.4096	0.64	0.00059375	2.28	6.12	130.64	5.1984	0.00001795152	0.0008743018	0.0000901720
0.7	0.2401	0.49	0.00034805	1.98	4.92	130.49	3.9204	0.00001965034	0.0010056158	0.0000915434
0.6	0.1296	0.36	0.00018787	1.72	3.88	130.36	2.9584	0.00002164546	0.0011564743	0.0000891265
0.5	0.0625	0.25	0.000090599	1.50	3.00	130.25	2.2500	0.00002441406	0.0013249715	0.0000813802
0.4	0.0256	0.16	0.000037109	1.36	2.28	130.16	1.8496	0.00002912555	0.0014603559	0.0000633583

	20	21	22	23	24	25	26	27	28
λ	λ^4(17+18+19)	H_1	$3+2\lambda^2$	[illegible]	$(3+2\lambda^2)(1+6\lambda^2)$	$(1+2\lambda^2)^3$	$(4+\lambda^2)$	$(2+\lambda^2)^2$	$5+2\lambda^2$
2.0	0.004426755	0.03082446	11.00	[illegible]	275.0000	729.0000	8.00	36.0000	13.00
1.9	0.003917245	0.02601273	10.22	22.66	231.5852	555.4122	7.61	31.4721	12.22
1.8	0.003440936	0.02186244	9.48	[illegible]	193.7712	418.5090	7.24	27.4576	11.48
1.7	0.002997142	0.01830857	8.78	18.34	161.0252	311.6658	6.89	23.9121	10.78
1.6	0.002585265	0.01528961	8.12	[illegible]	132.8432	229.2209	6.56	20.7936	10.12
1.5	0.002204803	0.01274767	7.50	14.50	108.7800	166.3750	6.25	18.0625	9.50
1.4	0.001856707	0.01062978	6.92	[illegible]	88.2992	118.3749	5.96	15.6816	8.92
1.3	0.001536650	0.00888116	6.38	[illegible]	71.0232	84.02567	5.69	13.6161	8.38
1.2	0.001248518	0.00745872	5.88	9.64	56.6832	58.41107	5.44	11.8336	7.88
1.1	0.000990927	0.00631761	[illegible]	[illegible]	44.7692	40.00169	5.21	10.3041	7.42
1.0	0.000763957	0.00541789	5.00	[illegible]	35.0000	27.00000	5.00	9.0000	7.00
0.9	0.000567764	0.00472318	4.62	[illegible]	27.0232	17.98473	4.81	7.8961	6.62
0.8	0.000402401	0.00420050	4.28	[illegible]	20.7152	11.85235	4.64	6.9696	6.28
0.7	0.000268146	0.00382054	3.98	[illegible]	15.6812	7.762392	4.49	6.2001	5.98
0.6	0.000164235	0.00355645	[illegible]	[illegible]	11.7552	5.088448	4.36	5.5696	5.72
0.5	0.0000894?3	0.00338437	[illegible]	2.50	8.7500	3.37500	4.25	5.0625	5.50
0.4	0.000039753	0.00328121	3.36	1.96	6.5856	2.515456	4.16	4.6656	5.36

	22	23	24	25	26	27	28	29	
λ	$-\frac{3}{64(1+2\lambda^2)}$	$-\frac{1}{384(2+\lambda^2)}$	$\lambda^4(22+23)$	H_2	$3+\lambda^4$	$(1+2\lambda^2)^3$	$4+11\lambda^2+10\lambda^4$	$1+3\lambda^2$	
2.0	-0.005208333	-0.000417361	-0.0900 1110	-0.2410528	19.0000	729.000	208.0000	13.00	
1.9	-0.005702555	-0.000464201	-0.08036578	-0.2268372	16.0321	555.4122	174.0310	11.83	
1.7	-0.006266711	-0.000496978	-0.07100250	-0.2131380	13.4976	418.5090	144.6160	10.72	
1.8	-0.006913717	-0.000532549	-0.06219196	-0.2002257	11.3521	311.6658	119.3110	9.67	
1.6	-0.007659314	-0.000571089	-0.05393877	-0.1881055	9.5536	229.2209	97.6960	8.68	
1.5	-0.008522727	-0.000612745	-0.04624833	-0.1767822	8.0625	166.3750	79.3750	7.75	
1.4	-0.009527439	-0.000657618	-0.03912691	-0.1662674	6.8416	118.3949	63.9760	6.88	
1.3	-0.010702055	-0.000705736	-0.03258179	-0.1565532	5.8561	84.02767	51.1510	6.07	
1.2	-0.012081186	-0.000757025	-0.02662131	-0.1476630	5.0736	58.41107	40.5760	5.32	
1.1	-0.013706140	-0.000811267	-0.02125494	-0.1396013	4.4641	40.00169	31.9510	4.63	
1.0	-0.015625000	-0.000868056	-0.01649306	-0.1323376	4.0000	27.00000	25.0000	4.00	
0.9	-0.017891221	-0.000926750	-0.01234647	-0.1260054	3.6561	17.98473	19.4710	3.43	
0.8	-0.020559211	-0.000986427	-0.00882509	-0.1204918	3.4096	11.85235	15.1360	2.92	
0.7	-0.023674242	-0.001045850	-0.00593529	-0.1158442	3.2401	7.762392	11.7710	2.47	
0.6	-0.027257907	-0.001103461	-0.00367499	-0.[illegible]20605	3.1296	5.088448	9.2560	2.08	
0.5	-0.031250000	-0.001157408	-0.00202546	-0.1091219	3.0625	3.375000	7.3750	1.75	
0.4	-0.034466912	-0.001205633	-0.00091322	-0.1069547	3.0256	2.515456	6.0160	1.48	

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H_3

λ	30 $\frac{3+\lambda^4}{4(1+2\lambda^2)^3}$	31 $-\frac{3(4+11\lambda^2+10\lambda^4)}{16(1+2\lambda^2)^2}$	32 $\frac{\pi^2}{24}\frac{1+3\lambda^2}{1+2\lambda^2}$	33 λ^2(30 + 31 + 32)	34 $-\frac{9}{16(1+2\lambda^2)^2}$	35 $\frac{\pi^2}{8}$ + 33 + 34		
2.0	+0.006515775	-0.481481481	+0.594003925	+0.476153068	-0.00785630	+1.7019974		
1.9	+0.007216307	-0.482930075	+0.591836026	+0.419201532	-0.00780827	+1.6450938		
1.8	+0.008062909	-0.484634605	+0.589361408	+0.361438667	-0.00893335	+1.5902059		
1.7	+0.009105988	-0.486656323	+0.586523374	+0.314931938	-0.01026916	+1.5383633		
1.6	+0.01041964	-0.48902415	+0.583252770	+0.267221546	-0.01186213	+1.4896100		
1.5	+0.01211495	-0.49199380	+0.579465415	+0.[illegible]406977	-0.01376955	+1.4440008		
1.4	+0.0144467	-0.498483211	+0.575058236	+0.178402561	-0.01615507	+1.3959481		
1.3	+0.01742313	-0.499927676	+0.569905814	+0.147701143	-0.01881692	+1.3625918		
1.2	+0.02171506	-0.505367202	+0.563856280	+0.115493944	-0.02213520	+1.3270593		
1.1	+0.02789945	-0.512192254	+0.556728420	+0.087646369	-0.02611772	+1.2952293		
1.0	+0.03703704	-0.520833333	+0.548311360	+0.064515067	-0.03086190	+1.2673538		
0.9	+0.05082228	-0.531847285	+0.538370601	+0.046449933	-0.03643151	+1.2437190		
0.8	+0.07191823	-0.545937211	+0.526667491	+0.033695046	-0.04280316	+1.2245924		
0.7	+0.10435250	-0.563425237	+0.513003432	+0.026181041	-0.05000794	+1.2098737		
0.6	+0.15376005	-0.586134667	+0.497305652	+0.023195173	-0.05686444	+1.2000314		
0.5	+0.22685165	-0.614583333	+0.47977244	+0.023010237	-0.06317561	+1.1935352		
0.4	+0.30070095	-0.609861592	+0.447518831	+0.022137310	-0.06358824	+1.1922497		

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λ	(36) $\frac{2}{9}(\frac{1}{\lambda^2}+\lambda^2)+\frac{4}{27}$	(37) $\frac{5\lambda^2}{9}H_1/\lambda^2$	(38) $8H_2/\lambda^2$	(39) H_3/λ^2	(40) (37) (39)	(41) (36) × (39)	(7) $K_0=2\sqrt{(41)}$
2.0	1.0925925	0.4383923	−0.4820056	0.4254993	0.1865356	0.4648973	1.36367
1.9	1.0119277	0.4099267	−0.5026613	0.4557047	0.1868055	0.4611402	1.35814
1.8	0.9367352	0.3838673	−0.5262117	0.4908043	0.1884037	0.4597537	1.35610
1.7	0.8450416	0.3603994	−0.5540565	0.5323056	0.1918426	0.4498204	1.34138
1.6	0.8038425	0.3397691	−0.5878277	0.5816789	0.1977045	0.4677390	1.36782
1.5	0.7469135	0.3223115	−0.6285551	0.6417781	0.2068525	0.4793527	1.38470
1.4	0.6970823	0.3085288	−0.6786220	0.7122164	0.2197377	0.4964748	1.40922
1.3	0.6551961	0.2989582	−0.7410802	0.8062173	0.2410402	0.5282632	1.45364
1.2	0.6224491	0.2946655	−0.8203500	0.9215190	0.2715546	0.5736482	1.51478
1.1	0.6006917	0.2970263	−0.9229838	1.0704374	0.3179481	0.6430027	1.60374
1.0	0.5925925	0.3082177	−1.0590288	1.2673538	0.3906209	0.7510244	1.73324
0.9	0.6024965	0.3317240	−1.2444778	1.5354556	0.5093425	0.9251066	1.92364
0.8	0.6375925	0.3733778	−1.5061475	1.9134256	0.7144306	1.2199858	2.20908
0.7	0.7105517	0.4435638	−1.8913337	2.4671300	1.0952167	1.754445	2.64910
0.6	0.8454320	0.5620069	−2.4902333	3.3334206	1.8734054	2.8181804	3.35748
0.5	1.0925925	0.7701322	−3.4919008	4.7741408	3.6767196	5.2161904	4.56778
0.4	1.5725925	1.1666524	−5.3477450	7.4515606	8.6933811	11.7182683	6.84638

C A B

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$$K = 2\left\{A\left(\frac{\delta}{t}\right)^2 + B\right\}^{\frac{1}{2}} - C\left(\frac{\delta}{t}\right)$$

$$2A\left(\frac{\delta}{t}\right) = C\left\{A\left(\frac{\delta}{t}\right)^2 + B\right\}^{\frac{1}{2}}$$

$$(4A^2 - C^2A)\left(\frac{\delta}{t}\right)^2 = C^2B$$

$$\left(\frac{\delta}{t}\right)^2 = \frac{C^2B}{4A^2 - C^2A}$$

$$\boxed{\left(\frac{\delta}{t}\right) = \sqrt{\frac{B}{\left(\frac{2A}{C}\right)^2 - A}}}$$

$$K_{min} = \left(\frac{4A}{C} - C\right)\sqrt{\frac{C^2B}{4A^2 - C^2A}}$$

$$\boxed{K_{min.} = \left[2\left(\frac{2A}{C}\right) - C\right]\sqrt{\frac{B}{\left(\frac{2A}{C}\right)^2 - A}}}$$

	(43)	(44)	(45)	(46)	(47)	(48)	(49)	(50)
λ	$2A/C$	$(43)^2 - A$	(41) ÷ (44)	$\frac{1}{t} = (45)^{\frac{1}{2}}$	2 (43) − C	$K_{min.}$		
2.0	0.7238371	0.4122883	1.127602					
1.9	0.7432289	0.3655837	1.261381					
1.8	0.7160008	0.3242534	1.417884					
1.7	0.6922496	0.2873669	1.565317					
1.6	0.6726591	0.2547658	1.835457					
1.5	0.6581802	0.2263487	2.117762					
1.4	0.6476062	0.1996539	2.486677					
1.3	0.6505104	0.1821236	2.900575					
1.2	0.6620457	0.1667499	3.440171					
1.1	0.6889571	0.1567138	4.103039					
1.0	0.7376965	0.1535752	4.890271	2.211396	0.9207461			
0.9	0.8185591	0.1606915	5.757035					
0.8	0.9486861	0.1855747	6.574095					
0.7	1.1581421	0.2460764	7.129676					
0.6	1.5046023	0.3904227	7.218380					
0.5	2.1058557	0.7579086	6.882348					
0.4	3.2512325	1.8771317	1.242646					

Mistake made !!!

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$$\frac{w}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left[1 - \left(\frac{x}{a}\right)^2 - \frac{f}{4}\left(1+\cos\frac{2\pi x}{a}\right)\left(1+\cos\frac{2\pi y}{a}\right)\right]$$

$$\frac{w_0}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left[1 - \left(\frac{x}{a}\right)^2\right]$$

$$R\frac{\partial^2 w}{\partial x^2} = \left[-1 + \frac{f}{2}\pi^2 \cos\frac{2\pi x}{a}\left(1+\cos\frac{2\pi y}{a}\right)\right]$$

$$R\frac{\partial^2 w}{\partial y^2} = \left[\; + \frac{f}{2}\pi^2 \left(1+\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi y}{a}\right]$$

$$R\frac{\partial^2 w}{\partial x \partial y} = \left[-\frac{f}{2}\pi^2 \sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a}\right]$$

$$R\frac{\partial^2 w_0}{\partial x^2} = [-1], \qquad R\frac{\partial^2 w_0}{\partial y^2} = R\frac{\partial^2 w_0}{\partial x\partial y} = 0$$

Thus

$$\nabla^4 F = \frac{E}{R^2}\left[\frac{f^2}{4}\pi^4 \sin^2\frac{2\pi x}{a}\sin^2\frac{2\pi y}{a}\right.$$

$$\left. + \frac{f}{2}\pi^2\left(1+\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi y}{a} - \frac{f^2}{4}\pi^4\left(1+\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi x}{a}\left(1+\cos\frac{2\pi y}{a}\right)\cos\frac{2\pi y}{a}\right]$$

$$= \frac{E}{R^2}\left[\frac{f}{2}\pi^2\left(1+\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi y}{a} + \frac{f^2}{4}\pi^4\left\{\frac{1}{4}\left(1-\cos\frac{4\pi x}{a}\right)\left(1-\cos\frac{4\pi y}{a}\right)\right.\right.$$

$$\left.\left. - \frac{1}{4}\left(2\cos\frac{2\pi x}{a} + 1 + \cos\frac{4\pi x}{a}\right)\left(2\cos\frac{2\pi y}{a} + 1 + \cos\frac{4\pi y}{a}\right)\right\}\right]$$

$$\nabla^4 F = \frac{E}{R^2}\Big[\frac{f}{2}\pi^2\left(1+\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi y}{a} + \frac{f^2}{16}\pi^4\Big\{\cancel{1} - 2\cos\frac{4\pi x}{a} - 2\cos\frac{4\pi y}{a} + \cancel{\cos\frac{4\pi x}{a}\cos\frac{4\pi y}{a}}$$

$$- 4\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{a} - 2\cos\frac{4\pi y}{a} - 2\cos\frac{4\pi x}{a}\cos\frac{2\pi y}{a} - 2\cos\frac{2\pi x}{a} - \cancel{1} - \cancel{\cos\frac{4\pi x}{a}}$$

$$- 2\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{a} - \cancel{\cos\frac{4\pi y}{a}} - \cancel{\cos\frac{4\pi x}{a}\cos\frac{4\pi y}{a}}\Big\}\Big]$$

$$= \frac{E}{R^2}\Big[\frac{f}{2}\pi^2\left(1+\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi y}{a} - \frac{f^2}{8}\pi^4\Big\{\cos\frac{2\pi x}{a} + \cos\frac{2\pi y}{a} + 2\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{a}$$

$$+ \cos\frac{4\pi x}{a} + \cos\frac{4\pi y}{a} + \cos\frac{4\pi x}{a}\cos\frac{2\pi y}{a} + \cos\frac{2\pi x}{a}\cos\frac{4\pi y}{a}\Big\}\Big]$$

$$\nabla^4 F = \frac{E}{R^2}\frac{(f\pi^2)}{2}\Big[-\frac{(f\pi^2)}{4}\cos\frac{2\pi x}{a} + \Big\{1 - \frac{(f\pi^2)}{4}\Big\}\cos\frac{2\pi y}{a} + \Big\{1 - \frac{f\pi^2}{2}\Big\}\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{a}$$

$$- \frac{f\pi^2}{4}\Big\{\cos\frac{4\pi x}{a} + \cos\frac{4\pi y}{a} + \cos\frac{4\pi x}{a}\cos\frac{2\pi y}{a} + \cos\frac{2\pi x}{a}\cos\frac{4\pi y}{a}\Big\}\Big]$$

$$\nabla^4 F = \frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = C\cos\frac{2\pi x}{a}, \qquad \text{Put } F = c\cos\frac{2\pi x}{a}$$

$$c\left\{\left(\frac{2\pi}{a}\right)^4 - \cancel{2\left(\frac{2\pi}{a}\right)^2}\right\} = C \qquad c = \frac{C}{\left(\frac{2\pi}{a}\right)^4 - \cancel{2\left(\frac{2\pi}{a}\right)^2}}$$

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Therefore the particular integral is

$$E\left(\frac{a}{R}\right)^2 \frac{f}{8}\left[-\frac{f\pi^2}{4}\,\frac{\cos\frac{2\pi x}{a}}{\left(\frac{2\pi}{a}\right)^2} + \left(1-\frac{f\pi^2}{4}\right)\frac{\cos\frac{2\pi y}{a}}{\left(\frac{2\pi}{a}\right)^2} + \frac{1}{4}\left(1-\frac{f\pi^2}{2}\right)\frac{\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{a}}{\left(\frac{2\pi}{a}\right)^2}\right.$$

$$\left.-\frac{f\pi^2}{4}\left\{\frac{1}{4}\,\frac{\cos\frac{4\pi x}{a}}{\left(\frac{4\pi}{a}\right)^2} + \frac{1}{4}\,\frac{\cos\frac{4\pi y}{a}}{\left(\frac{4\pi}{a}\right)^2} + \frac{1}{25}\,\frac{\cos\frac{4\pi x}{a}\cos\frac{2\pi y}{a}}{\left(\frac{2\pi}{a}\right)^2} + \frac{1}{25}\,\frac{\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{a}}{\left(\frac{2\pi}{a}\right)^2}\right\}\right]$$

The complementary function can be taken as

$$F = \frac{E}{\left(\frac{2\pi}{a}\right)^2}\left[A_2\cosh\left(\frac{2\pi y}{a}\right) + B_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right]\cos\frac{2\pi x}{a}$$

$$+\frac{E}{\left(\frac{4\pi}{a}\right)^2}\left[A_4\cosh\left(\frac{4\pi y}{a}\right) + B_4\left(\frac{4\pi y}{a}\right)\sinh\left(\frac{4\pi y}{a}\right)\right]\cos\frac{4\pi x}{a}$$

$$+\;E a^2\left(\frac{x}{a}\right)^2 C$$

$$\frac{\sigma_y}{E} = -\left[A_2 \cosh\left(\frac{2\pi y}{a}\right) + B_2\left(\frac{2\pi y}{a}\right)\sinh\frac{2\pi y}{a}\right]\cos\frac{2\pi x}{a} - \left[A_4 \cosh\left(\frac{4\pi y}{a}\right) + B_4\left(\frac{4\pi y}{a}\right)\sinh\left(\frac{4\pi y}{a}\right)\right]\cos\frac{4\pi x}{a}$$

$$+ 2C + \left(\frac{a}{R}\right)^2\frac{f}{8}\left[\frac{f\pi^2}{4}\cos\frac{2\pi x}{a} + \frac{f}{4}\left(\frac{f\pi^2}{2}-1\right)\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{a}\right.$$

$$\left. + \frac{f\pi^2}{4}\left\{\frac{f}{4}\cos\frac{4\pi x}{a} + \frac{4}{25}\cos\frac{4\pi x}{a}\cos\frac{2\pi y}{a} + \frac{1}{25}\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{a}\right\}\right]$$

The condition is $\frac{\sigma_y}{E} = -\frac{\sigma}{E}$, when $y = \pm a$.

Thus $-\frac{\sigma}{E} = 2C$ $\boxed{C = -\frac{\sigma}{2E}}$

$A_2 \cosh 2\pi + 2\pi B_2 \sinh 2\pi = \left(\frac{a}{R}\right)^2\frac{f}{8}\left[\frac{f\pi^2}{4} + \frac{f}{4}\left(\frac{f\pi^2}{2}-1\right) + \frac{f\pi^2}{4}\left(\frac{1}{25}\right)\right]$	
$A_4 \cosh 4\pi + 4\pi B_4 \sinh 4\pi = \left(\frac{a}{R}\right)^2\frac{f}{8}\left[\frac{f\pi^2}{4}\left(\frac{f}{4} + \frac{4}{25}\right)\right]$	
$(A_2+B_2)\sinh 2\pi + 2\pi B_2 \cosh 2\pi = 0$	from shear stress consideration
$(A_4+B_4)\sinh 4\pi + 4\pi B_4 \cosh 4\pi = 0$	

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$$\frac{\sigma_y}{E} = -\frac{\sigma}{E} + \left(\frac{a}{R}\right)^2 \frac{f}{8}\left[\left\{\frac{f\pi^2}{4} + \frac{f}{4}\left(\frac{f\pi^2}{2}-1\right)\cos\frac{2\pi y}{a} + \frac{f\pi^2}{100}\cos\frac{4\pi y}{a}\right\} - \left\{a_2\cosh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}\right]\cos\frac{2\pi x}{a}$$

$$+\left(\frac{a}{R}\right)^2 \frac{f}{8}\left[\left\{\frac{f\pi^2}{16} + \frac{f\pi^2}{25}\cos\frac{2\pi y}{a}\right\} - \left\{a_4\cosh\left(\frac{4\pi y}{a}\right) + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}\right]\cos\frac{4\pi x}{a}$$

$$\frac{1}{aE^2}\int_0^a \sigma_y^2\,dy = \left(\frac{\sigma}{E}\right)^2 + \left(\frac{a}{R}\right)^4\frac{f^2}{128}\left[\left\{\frac{f\pi^2}{4} + \frac{f}{4}\left(\frac{f\pi^2}{2}-1\right)\cos\frac{2\pi y}{a} + \frac{f\pi^2}{100}\cos\frac{4\pi y}{a}\right\} - \left\{a_2\cosh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}\right]^2$$

$$+\left(\frac{a}{R}\right)^4\frac{f^2}{128}\left[\left\{\frac{f\pi^2}{16} + \frac{f\pi^2}{25}\cos\frac{2\pi y}{a}\right\} - \left\{a_4\cosh\left(\frac{4\pi y}{a}\right) + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}\right]^2$$

$$\frac{1}{a^2E^2}\int_0^a\int_0^a \sigma_y^2\,dx\,dy = \left(\frac{\sigma}{E}\right)^2 + \left(\frac{a}{R}\right)^4\frac{f^2}{256}\left[2\left(\frac{f\pi^2}{4}\right)^2 + \frac{f}{16}\left(\frac{f\pi^2}{2}-1\right)^2 + \left(\frac{f\pi^2}{100}\right)^2 + 2\left(\frac{f\pi^2}{16}\right)^2 + \left(\frac{f\pi^2}{25}\right)^2\right]$$

$$-\left(\frac{a}{R}\right)^4\frac{f^2}{64}\left[\int_0^1\left\{\frac{f\pi^2}{4} + \frac{f}{4}\left(\frac{f\pi^2}{2}-1\right)\cos\frac{2\pi y}{a} + \frac{f\pi^2}{100}\cos\frac{4\pi y}{a}\right\}\left\{a_2\cosh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\} d\left(\frac{y}{a}\right)\right.$$

$$\left.+\int_0^1\left\{\frac{f\pi^2}{16} + \frac{f\pi^2}{25}\cos\frac{2\pi y}{a}\right\}\left\{a_4\cosh\left(\frac{4\pi y}{a}\right) + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\} d\left(\frac{y}{a}\right)\right]$$

$$+\left(\frac{a}{R}\right)^4\frac{f^2}{128}\left[\int_0^1\left\{a_2\cosh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}^2 d\left(\frac{y}{a}\right) + \int_0^1\left\{a_4\cosh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}^2 d\left(\frac{y}{a}\right)\right]$$

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$$\int_0^1 \left\{a_2 \cosh\left(\frac{2\pi y}{a}\right) + b_2 \left(\frac{2\pi y}{a}\right) \sinh\left(\frac{2\pi y}{a}\right)\right\} d\left(\frac{y}{a}\right) = \frac{1}{2\pi} \int_0^1 \left\{a_2 \cosh\left(\frac{2\pi y}{a}\right) + b_2 \left(\frac{2\pi y}{a}\right) \sinh\left(\frac{2\pi y}{a}\right)\right\} d\left(\frac{2\pi y}{a}\right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left\{a_2 \cosh u + b_2 u \sinh u\right\} du = \frac{1}{2\pi} \Big[a_2 \sinh u + b_2 (u \cosh u - \sinh u)\Big]_0^{2\pi}$$

$$= \frac{1}{2\pi} \Big[(a_2 - b_2) \sinh 2\pi + b_2\, 2\pi \cosh 2\pi \Big]$$

$$\int_0^1 \cos\frac{2\pi y}{a} \left\{ a_2 \cosh\left(\frac{2\pi y}{a}\right) + b_2 \left(\frac{2\pi y}{a}\right) \sinh\left(\frac{2\pi y}{a}\right)\right\} d\left(\frac{y}{a}\right)$$

$$= a_2 \int_0^1 \frac{1}{2} \left\{ \cosh \frac{2\pi}{a}(y+iy) + \cosh\frac{2\pi}{a}(y-iy)\right\} d\left(\frac{y}{a}\right) + b_2 \int_0^1 \frac{1}{2}\left(\frac{2\pi y}{a}\right)\left\{\sinh\frac{2\pi}{a}(y+iy) + \sinh\frac{2\pi}{a}(y-iy)\right\} d\left(\frac{y}{a}\right)$$

$$= \frac{a_2}{2}\frac{1}{2\pi}\left[\frac{1}{1+i} \sinh\frac{2\pi}{a}(y+iy) + \frac{1}{1-i}\sinh\frac{2\pi}{a}(y-iy)\right]_0^1$$

$$+ \frac{b_2}{2}\frac{1}{2\pi}\Bigg[\frac{1}{(1+i)^2}\left\{ \frac{2\pi y}{a}(1+i) \cosh\frac{2\pi}{a}(y+iy) - \sinh\frac{2\pi}{a}(y+iy)\right\}$$

$$+ \frac{1}{(1-i)^2}\left\{\frac{2\pi y}{a}(1-i)\cosh\frac{2\pi}{a}(y-iy) - \sinh\frac{2\pi}{a}(y-iy)\right\}\Bigg]_0^1$$

$$\int_0^1 \cos\frac{2\pi y}{a}\left\{a_2\cosh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\} d\left(\frac{y}{a}\right)$$

$$= \frac{a_2}{2}\frac{1}{2\pi}\left[\sinh\frac{2\pi y}{a}\cos\frac{2\pi y}{a} + \cosh\frac{2\pi y}{a}\sin\frac{2\pi y}{a}\right]_0^1$$

$$+ \frac{b_2}{2}\frac{1}{2\pi}\left[\left(\frac{2\pi y}{a}\right)\left\{\cosh\frac{2\pi y}{a}\cos\frac{2\pi y}{a} + \sinh\frac{2\pi y}{a}\sin\frac{2\pi y}{a}\right\} - \cosh\frac{2\pi y}{a}\sin\frac{2\pi y}{a}\right]_0^1$$

$$= \frac{a_2}{4\pi}\left[\sinh 2\pi\right] + \frac{b_2}{4\pi}\left[2\pi\cosh 2\pi\right]$$

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$$\int_0^1 \cos\frac{4\pi y}{a}\left\{a_2\cosh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\} d\left(\frac{y}{a}\right)$$

$$= \frac{a_2}{4\pi}\left[\frac{1}{1+2i}\sinh\frac{2\pi y}{a}(1+2i) + \frac{1}{1-2i}\sinh\frac{2\pi y}{a}(1-2i)\right]_0^1$$

$$+ \frac{b_2}{4\pi}\left[\frac{1}{(1+2i)^2}\left\{\frac{2\pi y}{a}(1+2i)\cosh\frac{2\pi y}{a}(1+2i) - \sinh\frac{2\pi y}{a}(1+2i)\right\}\right.$$

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$$\left. + \frac{1}{(1-2i)^2}\left\{\frac{2\pi y}{a}(1-2i)\cosh\frac{2\pi y}{a}(1-2i) - \sinh\frac{2\pi y}{a}(1-2i)\right\}\right]_0^1$$

$$- \frac{a_2}{4\pi}\left[\frac{2}{5}\sinh\frac{2\pi y}{a}\cos\frac{4\pi y}{a} + \frac{4}{5}\cosh\frac{2\pi y}{a}\sin\frac{4\pi y}{a}\right]_0^1$$

$$+ \frac{b_2}{4\pi}\left[\left(\frac{2\pi y}{a}\right)\left\{\frac{2}{5}\cosh\frac{2\pi y}{a}\cos\frac{4\pi y}{a} + \frac{4}{5}\sinh\frac{2\pi y}{a}\sin\frac{4\pi y}{a}\right\} + \left\{\frac{6}{25}\sinh\frac{2\pi y}{a}\cos\frac{4\pi y}{a} - \frac{8}{25}\cosh\frac{2\pi y}{a}\sin\frac{4\pi y}{a}\right\}\right]_0^1$$

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$$\int_0^1 \cos\frac{4\pi y}{a}\left\{a_2 \cosh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\} d\left(\frac{y}{a}\right)$$

$$= \frac{a_2}{4\pi}\,\frac{2}{5}\sinh 2\pi + \frac{b_2}{4\pi}\left[2\pi\,\frac{2}{5}\cosh 2\pi + \frac{6}{25}\sinh 2\pi\right]$$

$$\int_0^1 \left\{a_4 \cosh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\} d\left(\frac{y}{a}\right) = \frac{1}{4\pi}\left[(a_4 - b_4)\sinh 4\pi + b_4\, 4\pi \cosh 4\pi\right]$$

$$\int_0^1 \cos\frac{2\pi y}{a}\left\{a_4 \cosh\left(\frac{4\pi y}{a}\right) + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\} d\left(\frac{y}{a}\right)$$

$$= \frac{a_4}{2}\int_0^1\left[\cosh\frac{2\pi y}{a}(2+i) + \cosh\frac{2\pi y}{a}(2-i)\right] d\left(\frac{y}{a}\right) + \frac{b_4}{2}\int_0^1\left(\frac{2\pi y}{a}\right)\left[\sinh\frac{2\pi y}{a}(2+i) + \sinh\frac{2\pi y}{a}(2-i)\right] d\left(\frac{y}{a}\right)$$

$$= \frac{a_4}{2}\,\frac{1}{2\pi}\left[\frac{1}{2+i}\sinh\frac{2\pi y}{a}(2+i) + \frac{1}{2-i}\sinh\frac{2\pi y}{a}(2-i)\right]_0^1 + \frac{b_4}{2\pi}\left[\frac{1}{(2+i)^2}\left\{\frac{2\pi y}{a}(2+i)\cosh\frac{2\pi y}{a}(2+i) - \sinh\frac{2\pi y}{a}(2+i)\right\}\right.$$

$$\left. + \frac{1}{(2-i)^2}\left\{\frac{2\pi y}{a}(2-i)\cosh\frac{2\pi y}{a}(2-i) - \sinh\frac{2\pi y}{a}(2-i)\right\}\right]_0^1$$

$$= \frac{a_4}{4\pi}\left[\frac{4}{5}\sinh\frac{4\pi y}{a}\cos\frac{2\pi y}{a} + \frac{2}{5}\cosh\frac{4\pi y}{a}\sin\frac{2\pi y}{a}\right]_0^1 + \frac{b_4}{2\pi}\left[\left(\frac{2\pi y}{a}\right)\frac{4}{5}\cosh\frac{4\pi y}{a}\cos\frac{2\pi y}{a}\right.$$

$$\left. + \left(\frac{2\pi y}{a}\right)\frac{2}{5}\sinh\frac{4\pi y}{a}\sin\frac{2\pi y}{a} - \left\{\frac{6}{25}\sinh\frac{4\pi y}{a}\cos\frac{2\pi y}{a} + \frac{8}{25}\cosh\frac{4\pi y}{a}\sin\frac{2\pi y}{a}\right\}\right]_0^1$$

$$\int_0^1 \cos\frac{2\pi y}{a}\left\{a_4 \cosh\left(\frac{4\pi y}{a}\right) + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\} d\left(\frac{y}{a}\right)$$

$$= \frac{a_4}{4\pi}\frac{4}{5}\sinh 4\pi + \frac{b_4}{2\pi}\left[2\pi\frac{4}{5}\cosh 4\pi - \frac{6}{25}\sinh 4\pi\right]$$

$$\int_0^1 \left\{a_2 \cosh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}^2 d\left(\frac{y}{a}\right)$$

$$= \frac{a_2^2}{2}\int_0^1\left(\cosh\frac{4\pi y}{a}+1\right) d\left(\frac{y}{a}\right) + a_2 b_2\int_0^1\left(\frac{2\pi y}{a}\right)\sinh\frac{4\pi y}{a}\, d\left(\frac{y}{a}\right) + \frac{b_2^2}{2}\int_0^1\left(\frac{2\pi y}{a}\right)^2\left[\cosh\frac{4\pi y}{a}-1\right] d\left(\frac{y}{a}\right)$$

$$= \frac{a_2^2}{2}\left[\frac{1}{4\pi}\sinh\frac{4\pi y}{a} + \frac{y}{a}\right]_0^1 + \frac{a_2 b_2}{8\pi}\left[\left(\frac{4\pi y}{a}\right)\cosh\frac{4\pi y}{a} - \sinh\frac{4\pi y}{a}\right]_0^1$$

$$+ \frac{b_2^2}{2}\left[\frac{1}{16\pi}\left\{\left(\left(\frac{4\pi y}{a}\right)^2+2\right)\sinh\frac{4\pi y}{a} - \frac{8\pi y}{a}\cosh\frac{4\pi y}{a}\right\} - \frac{4\pi^2}{3}\left(\frac{y}{a}\right)^3\right]_0^1$$

$$= \frac{a_2^2}{2}\left[\frac{1}{4\pi}\sinh 4\pi + 1\right] + \frac{a_2 b_2}{8\pi}\left[4\pi\cosh 4\pi - \sinh 4\pi\right]$$

$$+ \frac{b_2^2}{2}\left[\frac{1}{16\pi}\left\{(16\pi^2+2)\sinh 4\pi - 8\pi\cosh 4\pi\right\} - \frac{4\pi^2}{3}\right]$$

$$\int_0^1 \left\{ a_4 \cosh\left(\frac{4\pi y}{a}\right) + b_4 \left(\frac{4\pi y}{a}\right) \sinh\left(\frac{4\pi y}{a}\right) \right\}^2 d\left(\frac{y}{a}\right)$$

$$= \frac{a_4^2}{2}\left[\frac{1}{8\pi}\sinh 8\pi + 1\right] + \frac{a_4 b_4}{16\pi}\left[8\pi\cosh 8\pi - \sinh 8\pi\right] + \frac{b_4^2}{2}\left[\frac{1}{32\pi}\left\{(64\pi^2+2)\sinh 8\pi - 16\pi\cosh 8\pi\right\} - \frac{16\pi^2}{3}\right]$$

$$\frac{1}{a^2E^2}\int_0^a\int_0^a \sigma_y^2\,dx\,dy = \left(\frac{a}{R}\right)^4 \frac{f^2}{256}\left[\frac{1}{8}(f\pi^2)^2 + \frac{1}{16}\left(\frac{f\pi^2}{2}-1\right)^2 + \frac{(f\pi^2)^2}{10{,}000} + \frac{1}{128}(f\pi^2)^2 + \frac{1}{625}(f\pi^2)^2\right]$$

$$- \left(\frac{a}{R}\right)^4 \frac{f^2}{64}\left[\frac{f\pi^2}{4}\left\{\frac{(a_2-b_2)}{2\pi}\sinh 2\pi + b_2\cosh 2\pi\right\} + \frac{1}{4}\left(\frac{f\pi^2}{2}-1\right)\left\{a_2\frac{\sinh 2\pi}{4\pi} + b_2\frac{\cosh 2\pi}{2}\right\}\right.$$

$$+ \frac{f\pi^2}{100}\left\{\frac{1}{10\pi}a_2\sinh 2\pi + b_2\left(\frac{1}{5}\cosh 2\pi + \frac{3}{50\pi}\sinh 2\pi\right)\right\}$$

$$\left. + \frac{f\pi^2}{16}\left\{\frac{(a_4-b_4)}{4\pi}\sinh 4\pi + b_2\cosh 4\pi\right\} + \frac{f\pi^2}{25}\left\{\frac{1}{5\pi}a_4\sinh 4\pi + b_4\left(\frac{4}{5}\cosh 4\pi - \frac{3}{25\pi}\sinh 4\pi\right)\right\}\right]$$

$$+ \left(\frac{a}{R}\right)^2\frac{f^2}{128}\left[\frac{a_2^2}{2}\left(\frac{\sinh 4\pi}{4\pi}+1\right) + \frac{a_2 b_2}{2}\left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right) + \frac{b_2^2}{2}\left(\frac{16\pi^2+2}{16\pi}\sinh 4\pi - \frac{1}{2}\cosh 4\pi - \frac{4\pi^2}{3}\right)\right.$$

$$\left. + \frac{a_4^2}{2}\left(\frac{\sinh 8\pi}{8\pi}+1\right) + \frac{a_4 b_4}{2}\left(\cosh 8\pi - \frac{\sinh 8\pi}{8\pi}\right) + \frac{b_4^2}{2}\left(\frac{64\pi^2+2}{32\pi}\sinh 8\pi - \frac{1}{2}\cosh 8\pi - \frac{16\pi^2}{3}\right)\right]$$

$$\frac{1}{E}\sigma_x = \left(\frac{a}{R}\right)^2 \frac{f}{8}\Bigg[\left\{(a_2+2b_2)\cosh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}\cos\frac{2\pi x}{a}$$

$$+\left\{(a_4+2b_4)\cosh\left(\frac{4\pi y}{a}\right) + b_4\left(\frac{4\pi y}{a}\right)\sinh\left(\frac{4\pi y}{a}\right)\right\}\cos\frac{4\pi x}{a}$$

$$+\left(\frac{f\pi^2}{4}-1\right)\cos\frac{2\pi y}{a} + \frac{1}{4}\left(\frac{f\pi^2}{2}-1\right)\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{a} + \frac{f\pi^2}{4}\Big\{\frac{1}{4}\cos\frac{4\pi y}{a} + \frac{1}{25}\cos\frac{4\pi x}{a}\cos\frac{2\pi y}{a}$$

$$+\frac{4}{25}\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{a}\Big\}\Bigg]$$

$$\frac{\sigma_x}{E} = \left(\frac{a}{R}\right)^2\frac{f}{8}\Bigg[\left\{\left(\frac{f\pi^2}{4}-1\right) + \frac{1}{4}\left(\frac{f\pi^2}{2}-1\right)\cos\frac{2\pi y}{a} + \frac{f\pi^2}{25}\cos\frac{4\pi y}{a} + (a_2+2b_2)\cosh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\sinh\frac{2\pi y}{a}\right\}\cos\frac{2\pi x}{a}$$

$$+\left\{\frac{1}{16}f\pi^2 + \frac{f\pi^2}{100}\cos\frac{2\pi y}{a} + (a_4+2b_4)\cosh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}\cos\frac{4\pi x}{a}\Bigg]$$

$$\frac{1}{a^2E^2}\int_0^a\int_0^a \sigma_x^2\,dx\,dy = \left(\frac{a}{R}\right)^4\frac{f^2}{256}\left[2\left(\frac{f\pi^2}{4}-1\right)^2 + \frac{1}{16}\left(\frac{f\pi^2}{2}-1\right)^2 + 2\left(\frac{f\pi^2}{16}\right)^2 + \frac{(f\pi^2)^2}{10{,}000} + \left(\frac{f\pi^2}{25}\right)^2\right]$$

$$+\left(\frac{a}{R}\right)^4\frac{f^2}{64}\Bigg[\int_0^1\left\{\left(\frac{f\pi^2}{4}-1\right)+\frac{1}{4}\left(\frac{f\pi^2}{2}-1\right)\cos\frac{2\pi y}{a} + \frac{f\pi^2}{25}\cos\frac{4\pi y}{a}\right\}\left\{(a_2+2b_2)\cosh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\sinh\frac{2\pi y}{a}\right\}d\left(\frac{y}{a}\right)$$

$$+\int_0^1\left\{\frac{1}{16}f\pi^2 + \frac{f\pi^2}{100}\right\}\left\{(a_4+2b_4)\cosh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}d\left(\frac{y}{a}\right)\Bigg]$$

$$+\left(\frac{a}{R}\right)^4\frac{f^2}{128}\left[\int_0^1\left\{(a_2+2b_2)\cosh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\sinh\frac{2\pi y}{a}\right\}^2 d\left(\frac{y}{a}\right) + \int_0^1\left\{(a_4+2b_4)\cosh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}^2 d\left(\frac{y}{a}\right)\right]$$

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$$\frac{1}{a^2E^2}\int_0^a\int_0^a \sigma_x^2\,dx\,dy = \left(\frac{a}{R}\right)^4 \frac{f^2}{256}\left[2\left(\frac{f\pi^2}{4}-1\right)^2 + \frac{1}{16}\left(\frac{f\pi^2}{2}-1\right)^2 + \frac{1}{128}(f\pi^2)^2 + \frac{(f\pi^2)^2}{10{,}000} + \left(\frac{f\pi^2}{25}\right)^2\right]$$

$$+\left(\frac{a}{R}\right)^4\frac{f^2}{64}\left[\left(\frac{f\pi^2}{4}-1\right)\left\{\frac{(a_2+b_2)}{2\pi}\sinh 2\pi + b_2\cosh 2\pi\right\} + \frac{1}{4}\left(\frac{f\pi^2}{2}-1\right)\left\{(a_2+2b_2)\frac{\sinh 2\pi}{4\pi} + b_2\frac{\cosh 2\pi}{2}\right\}\right.$$

$$+\frac{f\pi^2}{25}\left\{\frac{1}{10\pi}(a_2+2b_2)\sinh 2\pi + b_2\left(\frac{1}{5}\cosh 2\pi + \frac{3}{50\pi}\sinh 2\pi\right)\right\}$$

$$\left.+\frac{f\pi^2}{16}\left\{\frac{(a_4+b_4)}{4\pi}\sinh 4\pi + b_2\cosh 4\pi\right\} + \frac{f\pi^2}{100}\left\{\frac{1}{5\pi}(a_4+2b_4)\sinh 4\pi + b_4\left(\frac{4}{5}\cosh 4\pi - \frac{3}{25\pi}\sinh 4\pi\right)\right\}\right]$$

$$+\left(\frac{a}{R}\right)^2\frac{f^2}{128}\left[\frac{(a_2+2b_2)^2}{2}\left(\frac{\sinh 4\pi}{4\pi}+1\right) + \frac{(a_2+2b_2)b_2}{2}\left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right) + \frac{b_2^2}{2}\left(\frac{16\pi^2+2}{16\pi}\sinh 4\pi - \frac{1}{2}\cosh 4\pi - \frac{4\pi^2}{3}\right)\right.$$

$$\left.+\frac{(a_4+2b_4)^2}{2}\left(\frac{\sinh 8\pi}{8\pi}+1\right) + \frac{(a_2+2b_2)b_2}{2}\left(\cosh 8\pi - \frac{\sinh 8\pi}{8\pi}\right) + \frac{b_2^2}{2}\left(\frac{64\pi^2+2}{32\pi}\sinh 8\pi - \frac{1}{2}\cosh 8\pi - \frac{16\pi^2}{3}\right)\right]$$

$$\frac{1}{E}\tau_{xy} = \left(\frac{a}{R}\right)^2 \frac{f}{8}\Bigg[\frac{1}{4}\left(\frac{f\pi^2}{2}-1\right) \sin\frac{2\pi x}{a} \sin\frac{2\pi y}{a} + \frac{f\pi^2}{4}\left\{ \frac{2}{25} \sin\frac{4\pi x}{a} \sin\frac{2\pi y}{a} + \frac{2}{25} \sin\frac{2\pi x}{a} \sin\frac{4\pi y}{a} \right\}$$

$$+ \left\{ (a_2+b_2) \sinh\frac{2\pi y}{a} + b_2 \left(\frac{2\pi y}{a}\right) \cosh\frac{2\pi y}{a} \right\} \sin\frac{2\pi x}{a} + \left\{ (a_4+b_4) \sinh\frac{4\pi y}{a} + b_4 \left(\frac{4\pi y}{a}\right) \cosh\frac{4\pi y}{a} \right\} \sin\frac{4\pi x}{a} \Bigg]$$

$$\frac{1}{aE^2}\int_0^a \tau_{xy}^2\, dx = \left(\frac{a}{R}\right)^4 \frac{f^2}{128}\Bigg[\left\{ \frac{1}{4}\left(\frac{f\pi^2}{2}-1\right) \sin\frac{2\pi y}{a} + (a_2+b_2) \sinh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right) \cosh\frac{2\pi y}{a} \right\}^2$$

$$\frac{f\pi^2}{50} \sin\frac{4\pi y}{a}$$

$$+ \left\{ \frac{f\pi^2}{50} \sin\frac{2\pi y}{a} + (a_4+b_4) \sinh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right) \cosh\frac{4\pi y}{a} \right\}^2 \Bigg]$$

$$\frac{1}{a^2E^2}\int_0^a\int_0^a \tau_{xy}^2\, dx\, dy = \left(\frac{a}{R}\right)^4 \frac{f^2}{256}\left[\frac{1}{16}\left(\frac{f\pi^2}{2}-1\right)^2 + 2\left(\frac{f\pi^2}{50}\right)^2 \right]$$

$$+ \left(\frac{a}{R}\right)^4 \frac{f^2}{64}\Bigg[\int_0^1 \left\{ \frac{1}{4}\left(\frac{f\pi^2}{2}-1\right) \sin\frac{2\pi y}{a} + \frac{f\pi^2}{50} \sin\frac{4\pi y}{a} \right\}\left\{ (a_2+b_2) \sinh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right) \cosh\frac{2\pi y}{a} \right\} d\left(\frac{y}{a}\right)$$

$$+ \frac{1}{50} f\pi^2 \int_0^1 \sin\frac{2\pi y}{a} \left\{ (a_4+b_4) \sinh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right) \cosh\frac{4\pi y}{a} \right\} d\left(\frac{y}{a}\right) \Bigg]$$

$$+ \left(\frac{a}{R}\right)^4 \frac{f^2}{128}\left[\int_0^1 \left\{ (a_2+b_2) \sinh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right) \cosh\frac{2\pi y}{a} \right\}^2 d\left(\frac{y}{a}\right) + \int_0^1 \left\{ (a_4+b_4) \sinh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right) \cosh\frac{4\pi y}{a} \right\}^2 d\left(\frac{y}{a}\right) \right]$$

$$\int_0^1 \sin\frac{2\pi y}{a}\sinh\frac{2\pi y}{a}\, d\left(\frac{y}{a}\right) = \frac{1}{2i}\int_0^1 \left[\cosh\frac{2\pi y}{a}(1+i) - \cosh\frac{2\pi y}{a}(1-i)\right] d\left(\frac{y}{a}\right)$$

$$= \frac{1}{4\pi i}\left[\frac{1}{1+i}\sinh\frac{2\pi y}{a}(1+i) - \frac{1}{1-i}\sinh\frac{2\pi y}{a}(1-i)\right]_0^1 = \frac{1}{4\pi}\left[-\sinh\frac{2\pi y}{a}\cos\frac{2\pi y}{a} + \cosh\frac{2\pi y}{a}\sin\frac{2\pi y}{a}\right]_0^1$$

$$= -\frac{\sinh 2\pi}{4\pi}$$

$$\int_0^1 \sin\frac{2\pi y}{a}\left(\frac{2\pi y}{a}\right)\cosh\frac{2\pi y}{a}\, d\left(\frac{y}{a}\right) = \frac{1}{2i}\int_0^1 \left(\frac{2\pi y}{a}\right)\left[\sinh\frac{2\pi y}{a}(1+i) - \sinh\frac{2\pi y}{a}(1-i)\right] d\left(\frac{y}{a}\right)$$

$$= \frac{1}{4\pi i}\left[\frac{1}{(1+i)^2}\left\{\frac{2\pi y}{a}(1+i)\cosh\frac{2\pi y}{a}(1+i) - \sinh\frac{2\pi y}{a}(1+i)\right\} - \frac{1}{(1-i)^2}\left\{\frac{2\pi y}{a}(1-i)\cosh\frac{2\pi y}{a}(1-i) - \sinh\frac{2\pi y}{a}(1-i)\right\}\right]_0^1$$

$$= \frac{1}{4\pi}\left[\frac{2\pi y}{a}\left\{-\cosh\frac{2\pi y}{a}\cos\frac{2\pi y}{a} + \sinh\frac{2\pi y}{a}\sin\frac{2\pi y}{a}\right\} + \sinh\frac{2\pi y}{a}\cos\frac{2\pi y}{a}\right]_0^1$$

$$= \left[-\frac{1}{2}\cosh 2\pi + \frac{\sinh 2\pi}{4\pi}\right]$$

$$\int_0^1 \sin\frac{4\pi y}{a}\sinh\frac{2\pi y}{a}\, d\left(\frac{y}{a}\right) = \frac{1}{4\pi i}\left[\frac{1}{1+2i}\sinh\frac{2\pi y}{a}(1+2i) - \frac{1}{1-2i}\sinh\frac{2\pi y}{a}(1-2i)\right]_0^1$$

$$= \frac{1}{4\pi}\left[-\frac{4}{5}\sinh\frac{2\pi y}{a}\cos\frac{4\pi y}{a} + \frac{2}{5}\cosh\frac{2\pi y}{a}\sin\frac{4\pi y}{a}\right]_0^1 = -\frac{\sinh 2\pi}{5\pi}$$

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$$\int_0^1 \sin\frac{2\pi y}{a}\,\sinh\frac{4\pi y}{a}\,d\left(\frac{y}{a}\right) = \frac{1}{4\pi i}\left[\frac{1}{2+i}\sinh\frac{2\pi y}{a}(2+i) - \frac{1}{2-i}\sinh\frac{2\pi y}{a}(2-i)\right]_0^1$$

$$= \frac{1}{4\pi}\left[-\frac{2}{5}\sinh\frac{4\pi y}{a}\cos\frac{2\pi y}{a} + \frac{4}{5}\cosh\frac{4\pi y}{a}\sin\frac{2\pi y}{a}\right]_0^1 = -\frac{\sinh 4\pi}{10\pi}$$

$$\int_0^1 \frac{4\pi y}{a}\sin\frac{2\pi y}{a}\cosh\frac{4\pi y}{a}\,d\left(\frac{y}{a}\right) = \frac{1}{2\pi i}\left[\frac{1}{(2+i)^2}\left\{\frac{2\pi y}{a}(2+i)\cosh\frac{2\pi y}{a}(2+i) - \sinh\frac{2\pi y}{a}(2+i)\right\}\right.$$

$$\left. - \frac{1}{(2-i)^2}\left\{\frac{2\pi y}{a}(2-i)\cosh\frac{2\pi y}{a}(2-i) - \sinh\frac{2\pi y}{a}(2-i)\right\}\right]_0^1$$

$$= \frac{1}{2\pi}\left[\frac{2\pi y}{a}\left\{-\frac{2}{5}\cosh\frac{4\pi y}{a}\cos\frac{2\pi y}{a} + \frac{4}{5}\sinh\frac{4\pi y}{a}\sin\frac{2\pi y}{a}\right\} + \left\{\frac{8}{25}\sinh\frac{4\pi y}{a}\cos\frac{2\pi y}{a}\right.\right.$$

$$\left.\left. - \frac{6}{25}\cosh\frac{4\pi y}{a}\sin\frac{2\pi y}{a}\right\}\right]_0^1$$

$$= -\frac{2}{5}\cosh 4\pi + \frac{4}{25\pi}\sinh 4\pi$$

$$\int_0^1 \left\{(a_2+b_2)\sinh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\cosh\frac{2\pi y}{a}\right\}^2 d\left(\frac{y}{a}\right)$$

$$= \frac{(a_2+b_2)^2}{2\pi}\left\{\frac{\sinh 4\pi}{4} - \pi\right\} + \frac{b_2(a_2+b_2)}{8\pi}\left\{4\pi\cosh 4\pi - \sinh 4\pi\right\} + \frac{b_2^2}{32\pi}\left[\frac{64\pi^3}{3} + (16\pi^2+2)\sinh 4\pi - 8\pi\cosh 4\pi\right]$$

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$$\int_0^1 \left\{ (a_4+b_4)\sinh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\cosh\frac{4\pi y}{a} \right\}^2 d\left(\frac{y}{a}\right)$$

$$= \frac{(a_4+b_4)^2}{4\pi}\left\{\frac{\sinh 8\pi}{4} - 2\pi\right\} + \frac{b_4(a_4+b_4)}{16\pi}\left\{8\pi\cosh 8\pi - \sinh 8\pi\right\} + \frac{b_4^2}{64\pi}\left[\frac{(8\pi)^3}{3} + (64\pi^2+2)\sinh 8\pi - 16\pi\cosh 8\pi\right]$$

$$\frac{1}{a^2E^2}\int_0^a\int_0^a \tau_{xy}^2\,dx\,dy = \left(\frac{a}{R}\right)^4\frac{f^2}{256}\left[\frac{1}{16}\left(\frac{f\pi^2}{3}-1\right)^2 + 2\left(\frac{f\pi^2}{50}\right)^2\right]$$

$$+\left(\frac{a}{R}\right)^4\frac{f^2}{64}\left[\frac{1}{4}\left(\frac{f\pi^2}{9}-1\right)\left\{-(a_2+b_2)\frac{\sinh 2\pi}{4\pi} + b_2\left(\frac{\sinh 2\pi}{4\pi} - \frac{1}{2}\cosh 2\pi\right)\right\}\right.$$

$$+\frac{f\pi^2}{50}\left\{-(a_2+b_2)\frac{\sinh 2\pi}{5\pi} + b_2\left(-\frac{2}{5}\cosh 2\pi + \frac{2}{25\pi}\sinh 2\pi\right)\right\}$$

$$\left.+\frac{f\pi^2}{50}\left\{-(a_4+b_4)\frac{\sinh 4\pi}{10\pi} + b_4\left(-\frac{2}{5}\cosh 4\pi + \frac{4}{25\pi}\sinh 4\pi\right)\right\}\right]$$

$$+\left(\frac{a}{R}\right)^4\frac{f^2}{128}\left[\frac{(a_2+b_2)^2}{2}\left\{\frac{\sinh 4\pi}{4\pi} - 1\right\} + \frac{b_2(a_2+b_2)}{2}\left\{\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right\} + \frac{b_2^2}{2}\left\{\frac{4\pi^2}{3} + \frac{16\pi^2+2}{16\pi}\sinh 4\pi - \frac{\cosh 4\pi}{2}\right\}\right.$$

$$\left.+\frac{(a_4+b_4)^2}{2}\left\{\frac{\sinh 8\pi}{8\pi} - 1\right\} + \frac{b_2(a_2+b_2)}{2}\left\{\cosh 8\pi - \frac{\sinh 8\pi}{8\pi}\right\} + \frac{b_2^2}{2}\left\{\frac{16\pi^2}{3} + \frac{64\pi^2+2}{32\pi}\sinh 8\pi - \frac{\cosh 8\pi}{2}\right\}\right]$$

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$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial y}\right)^2\right\} = \frac{1}{2}\left(\frac{a}{R}\right)^4\left[\frac{f}{4}\left(1+\cos\frac{2\pi x}{a}\right)\left(\frac{\pi}{a}\right)\sin\frac{2\pi y}{a}\right]^2 R^2$$

$$= \frac{1}{2}\left(\frac{a}{R}\right)^2\frac{f}{4}\left[\frac{f\pi^2}{4}\left(1+\cos\frac{2\pi x}{a}\right)^2\sin^2\frac{2\pi y}{a}\right] = \left(\frac{a}{R}\right)^2\frac{f}{8}\left[\frac{f\pi^2}{4}\left(\frac{3}{2}+2\cos\frac{2\pi x}{a}+\frac{1}{2}\cos\frac{4\pi x}{a}\right)\left(\frac{1}{2}-\frac{1}{2}\cos\frac{4\pi y}{a}\right)\right]$$

$$\frac{\partial v}{\partial y} = \frac{\sigma_y}{E} - \frac{1}{2}\left\{\left(\frac{\partial w}{\partial y}\right)^2\right\}$$

$$= -\frac{\sigma}{E} + \left(\frac{a}{R}\right)^2\frac{f}{8}\left[-\frac{3}{16}f\pi^2 + \frac{3}{16}f\pi^2\cos\frac{4\pi y}{a}\right) + \left[\left\{\frac{f}{4}\left(\frac{f\pi^2}{2}-1\right)\cos\frac{2\pi y}{a} + \frac{26}{100}f\pi^2\cos\frac{4\pi y}{a}\right\} - \left\{a_2\cosh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}\right]\cos\frac{2\pi x}{a}$$

$$+\left(\frac{a}{R}\right)^2\frac{f}{8}\left[\left\{\frac{f\pi^2}{25}\cos\frac{2\pi y}{a} + \frac{f\pi^2}{16}\cos\frac{4\pi y}{a}\right\} - \left\{a_4\cosh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}\right]\cos\frac{4\pi x}{a}$$

$$\frac{v}{a} = -\frac{\sigma}{E}\left(\frac{y}{a}\right) + \left(\frac{a}{R}\right)^2\frac{f}{8}\left[-\frac{3}{16}f\pi^2\left(\frac{y}{a}\right) + \frac{3}{64\pi}f\pi^2\sin\frac{4\pi y}{a} + \right.$$

$$\left\{\frac{f}{8\pi}\left(\frac{f\pi^2}{2}-1\right)\sin\frac{2\pi y}{a} + \frac{26}{400\pi}f\pi^2\sin\frac{4\pi y}{a}\right\}\cos\frac{2\pi x}{a} - \frac{1}{2\pi}\left\{(a_2-b_2)\sinh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\cosh\left(\frac{2\pi y}{a}\right)\right\}\cos\frac{2\pi x}{a}$$

$$+\left(\frac{a}{R}\right)^2\frac{f}{8}\left[\left\{\frac{f\pi^2}{50\pi}\sin\frac{2\pi y}{a} + \frac{f\pi^2}{64\pi}\sin\frac{4\pi y}{a}\right\} - \frac{1}{4\pi}\left\{(a_4-b_4)\sinh\left(\frac{4\pi y}{a}\right) + b_4\left(\frac{4\pi y}{a}\right)\cosh\left(\frac{4\pi y}{a}\right)\right\}\right]\cos\frac{4\pi x}{a}$$

At $y = \pm a$,

$$\left(\frac{v}{a}\right)_a = -\frac{\sigma}{E} + \left(\frac{a}{R}\right)^2 \frac{f}{8}\Big[-\frac{3}{16} f\pi^2 - \frac{1}{2\pi}\left\{(a_2 - b_2)\sinh 2\pi + 2\pi b_2 \cosh 2\pi\right\}\cos\frac{2\pi x}{a}$$
$$- \frac{1}{4\pi}\left\{(a_4 - b_4)\sinh 4\pi + 4\pi b_4 \cosh 4\pi\right\}\cos\frac{4\pi x}{a}\Big]$$

The potential increase $\Delta\wp$ is

$$at\int_0^a \sigma\left(\frac{v}{a}\right)_a dx \quad , \quad \text{or} \quad \boxed{\frac{\Delta\wp}{a^2 t E} = -\left(\frac{a}{R}\right)^2 \frac{f}{8}\frac{\sigma}{E}\left(\frac{3}{16} f\pi^2\right)} \qquad \left[-\left(\frac{\sigma}{E}\right)^2 \right.$$

cancel with term in σ_y energy]

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w_0}{\partial x^2} = \frac{1}{R}\left[\frac{f}{2}\pi^2 \cos\frac{2\pi x}{a}\left(1 + \cos\frac{2\pi y}{a}\right)\right]$$

$$\frac{t^2}{12a^2}\int_0^a\int_0^a\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w_0}{\partial x^2}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{2}\right)^2 \int_0^a\int_0^a\left\{\cos\frac{2\pi x}{a}\left(1+\cos\frac{2\pi y}{a}\right)\right\}^2 d\left(\frac{x}{a}\right) d\left(\frac{y}{a}\right)$$

$$= \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{2}\right)^2 \frac{1}{4}(2+1) = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{2}\right)^2\frac{3}{4}$$

$$\frac{t^2}{12a^2}\int_0^a\int_0^a\left(\frac{\partial^2 w}{\partial y^2}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{2}\right)^2\frac{3}{4}$$

$$\frac{t^2}{12a^2}\int_0^a\int_0^a\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{2}\right)^2\frac{1}{4}$$

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$$\cosh 2\pi \cdot a_2 + 2\pi \sinh 2\pi \cdot b_2 = \frac{77}{200}(f\pi^2) - \frac{1}{4}$$

$$\sinh 2\pi \cdot a_2 + (\sinh 2\pi + 2\pi \cosh 2\pi)\, b_2 = 0$$

$$2(\sinh 4\pi + 4\pi)\, b_2 = \sinh 2\pi \left(1 - \frac{77}{50} f\pi^2\right)$$

$$\boxed{b_2 = \frac{\sinh 2\pi}{2(\sinh 4\pi + 4\pi)} \left(1 - \frac{77}{50} f\pi^2\right)}$$

$$a_2 = - b_2 \frac{\sinh 2\pi + 2\pi \cosh 2\pi}{\sinh 2\pi}$$

$$\boxed{a_2 = - \frac{\sinh 2\pi + 2\pi \cosh 2\pi}{2(\sinh 4\pi + 4\pi)} \left(1 - \frac{77}{50} f\pi^2\right)}$$

$$\cosh 4\pi\ a_4 + 4\pi \sinh 4\pi \cdot b_4 = \frac{41}{400}(f\pi^2)$$

$$\sinh 4\pi\ a_4 + (\sinh 4\pi + 4\pi \cosh 4\pi)\, b_4 = 0$$

$$\frac{1}{2}(\sinh 8\pi + 8\pi) b_4 = -\frac{41}{400}(f\pi^2) \sinh 4\pi$$

$$\boxed{b_4 = - \frac{\sinh 4\pi}{(\sinh 8\pi + 8\pi)} \frac{41}{200}(f\pi^2)}$$

$$\boxed{a_4 = + \frac{\sinh 4\pi + 4\pi \cosh 4\pi}{(\sinh 8\pi + 8\pi)} \frac{41}{200}(f\pi^2)}$$

The extensional energy $= \frac{1}{2E}\left[\sigma_x^2 + \sigma_y^2 + 2\tau_{xy}^2 \right]$

$$2\,\frac{\text{The extensional energy}}{Ea^2t} = \left(\frac{a}{R}\right)^4 \frac{f^2}{64}\left[\frac{1}{32}(f\pi^2)^2 + \frac{1}{2}\left(\frac{f\pi^2}{4}-1\right)^2 + \frac{1}{16}\left(\frac{f\pi^2}{2}-1\right)^2 + \frac{(f\pi^2)^2}{20000} + \frac{1}{256}(f\pi^2)^2 + \frac{3}{2500}(f\pi^2)^2\right]$$

$$+\left(\frac{a}{R}\right)^4 \frac{f^2}{64}\left[-a_2\,\frac{3}{4}\,\frac{\sinh 2\pi}{2\pi}\left\{1+\frac{9}{50}(f\pi^2)\right\} - b_2\left\{\left(\frac{f}{4}\,\frac{\sinh 2\pi}{2\pi} + \frac{3}{4}\cosh 2\pi\right) + \left(\frac{139}{1000}\cosh 2\pi - \frac{3123}{5000}\,\frac{\sinh 2\pi}{2\pi}\right)(f\pi^2)\right\}\right.$$

$$\left. - a_4\,\frac{f\pi^2}{25}\,\frac{\sinh 4\pi}{4\pi} - b_4\left\{\quad \cdot \cosh 4\pi - \frac{33}{8}\,\frac{\sinh 4\pi}{4\pi}\right\}\frac{(f\pi^2)}{25}\right]$$

$$+\left(\frac{a}{R}\right)^2 \frac{f^2}{64}\left[\quad \frac{\sinh 4\pi}{4\pi}\,a_2^2 + \left(\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi\right) a_2 b_2 + \frac{1}{2}\left\{(8\pi^2+2)\,\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi + 1\right\} b_2^2\right.$$

$$\left. + \frac{\sinh 8\pi}{8\pi}\,a_4^2 + \left(\frac{\sinh 8\pi}{8\pi} + \cosh 8\pi\right) a_4 b_4 + \frac{1}{2}\left\{(32\pi^2+2)\,\frac{\sinh 8\pi}{8\pi} + \cosh 8\pi + 1\right\} b_4^2\right]$$

$$2\,\frac{\Delta p}{Ea^2t} = -\left(\frac{a}{R}\right)^2 \frac{f^2}{64}\left[3\pi^2\,\frac{\sigma}{E}\right]$$

$$2\,\frac{\text{The bending energy}}{Ea^2t} = \frac{1}{6}\left(\frac{t}{R}\right)^2 \frac{(f\pi^2)^2}{4}$$

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a_2^2	$a_2 b_2$	b_2^2
$\frac{\sinh 4\pi}{4\pi}+1$ λ^4	$4\frac{\sinh 4\pi}{4\pi}+4$ $\cosh 4\pi-\frac{\sinh 4\pi}{4\pi}$ λ^4	$4\frac{\sinh 4\pi}{4\pi}+4$ $2\cosh 4\pi-2\frac{\sinh 4\pi}{4\pi}$ λ^0 $\frac{16\pi^2+2}{16\pi}\sinh 4\pi-\frac{1}{2}\cosh 4\pi-\frac{4\pi^2}{3}$
$\frac{\sinh 4\pi}{4\pi}+1$	$\cosh 4\pi-\frac{\sinh 4\pi}{4\pi}$	$\frac{16\pi^2+2}{16\pi}\sinh 4\pi-\frac{1}{2}\cosh 4\pi-\frac{4\pi^2}{3}$
$2\frac{\sinh 4\pi}{4\pi}-2$ λ^2	$4\frac{\sinh 4\pi}{4\pi}-4$ $2\cosh 4\pi-2\frac{\sinh 4\pi}{4\pi}$ λ^2	$2\frac{\sinh 4\pi}{4\pi}-2$ $2\cosh 4\pi-2\frac{\sinh 4\pi}{4\pi}$ $2\frac{4\pi^2}{3}-\cosh 4\pi+2\frac{16\pi^2+2}{16\pi}\sinh 4\pi$ λ^2
$4\frac{\sinh 4\pi}{4\pi}$	$4\frac{\sinh 4\pi}{4\pi}+4\cosh 4\pi+$	$(16\pi^2+4)\frac{\sinh 4\pi}{4\pi}+2\cosh 4\pi+2$

$2\frac{1}{2}\lambda^4+\frac{1}{2}+\lambda^2$

$$(f\pi^2)^2\left[-\frac{1}{32}+\frac{1}{32}+\frac{1}{64}+\frac{1}{20{,}000}+\frac{1}{256}+\frac{3}{2500}\right]=(f\pi^2)^2\left[0.08203125+0.00125000\right]$$

$$+(f\pi^2)\left[-\frac{1}{4}-\frac{1}{16}\right]=-(f\pi^2)\left[0.3125000\right]$$

$$+\left[\frac{1}{2}+\frac{1}{16}\right]=0.5625000$$

$$2W_1=\left(\frac{a}{R}\right)^4\frac{f^2}{64}\Big[0.08328125(f\pi^2)^2-0.3125000(f\pi^2)+0.5625000$$

$$-0.75\,\frac{\sinh 2\pi}{2\pi}\left\{1+0.18000(f\pi^2)\right\}a_2-\left\{\left(1.2500\,\frac{\sinh 2\pi}{2\pi}+0.7500\cosh 2\pi\right)+\left(0.13900\cosh 2\pi-0.624600\,\frac{\sinh 2\pi}{2\pi}\right)(f\pi^2)\right\}b_2$$

$$-0.0400\,\frac{\sinh 4\pi}{4\pi}(f\pi^2)\,a_4-\left\{0.0400\cosh 4\pi-0.165000\,\frac{\sinh 4\pi}{4\pi}\right\}(f\pi^2)\,b_4$$

$$+\frac{\sinh 4\pi}{4\pi}a_2^2+\left(\frac{\sinh 4\pi}{4\pi}+\cosh 4\pi\right)a_2b_2+\left\{(4\pi^2+1)\frac{\sinh 4\pi}{4\pi}+0.5\cosh 4\pi+0.5\right\}b_2^2+$$

$$+\frac{\sinh 8\pi}{8\pi}a_4^2+\left(\frac{\sinh 8\pi}{8\pi}+\cosh 8\pi\right)a_4b_4+\left\{(16\pi^2+1)\frac{\sinh 8\pi}{8\pi}+0.5\cosh 8\pi+0.5\right\}b_4^2\Big]$$

$$2W_2=-\left(\frac{a}{R}\right)^2\frac{f^2}{64}\left[3\pi^2\frac{\sigma}{E}\right]$$

$$2W_3=\left(\frac{t}{R}\right)^2\frac{f^2}{64}\left[\frac{16\pi^4}{6}\right]$$

$$\boxed{\begin{aligned}
a_2 &= -\frac{\left(\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi\right)}{\left(\frac{\sinh 4\pi}{4\pi} + 1\right)}\left(0.25000 - 0.385000 f\pi^2\right) \\[2ex]
b_2 &= +\frac{\frac{\sinh 2\pi}{2\pi}}{\left(\frac{\sinh 4\pi}{4\pi} + 1\right)}\left(0.25000 - 0.385000 f\pi^2\right) \\[2ex]
a_4 &= +\frac{\left(\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi\right)}{\left(\frac{\sinh 8\pi}{8\pi} + 1\right)}\, 0.1025000(f\pi^2) \\[2ex]
b_4 &= -\frac{\left(\frac{\sinh 4\pi}{4\pi}\right)}{\left(\frac{\sinh 8\pi}{8\pi} + 1\right)}\, 0.102500(f\pi^2)
\end{aligned}}$$

$$\log_e(e^\xi) = \xi = 2.30258509 \log_{10}(e^\xi)$$

$$\log_{10}(e^\xi) = 0.434294482\,\xi$$

$$\therefore \log_{10}(e^{2\pi}) = 0.434294482 \cdot 6.283185307 = 2.7288[illegible]$$

$$e^{2\pi} = 535.49162, \qquad e^{-2\pi} = 0.00187$$

$$\sinh 2\pi = 267.74675 \qquad \cosh 2\pi = 267.74862$$

$$\boxed{\frac{\sinh 2\pi}{2\pi} = 42.613218 \qquad \cosh 2\pi = 267.74862}$$

$$\log_{10}\left(e^{4\pi}\right) = 5.45750542,$$

$$e^{4\pi} = 286751.33 \qquad \sinh 4\pi = 143375.66$$

$$\boxed{\frac{\sinh 4\pi}{4\pi} = 11409.473\,, \qquad \cosh 4\pi = 143375.66}$$

$$\log_{10}\left(e^{8\pi}\right) = 10.91501083\,, \quad e^{8\pi} = 82226314000$$

$$\boxed{\frac{\sinh 8\pi}{8\pi} = 1635840500 \qquad \cosh 8\pi = 41113157000}$$

$$\boxed{\begin{aligned} a_2 &= -\,0.027199735\,(0.2500 - 0.38500\,f\pi^2) \\ b_2 &= +\,0.0037345707\,(0.2500 - 0.38500\,f\pi^2) \\ a_4 &= +\,0.000094621164\,(0.1025000 - f\pi^2) \\ b_4 &= -\,0.0000089746855\,(0.1025000 - f\pi^2) \end{aligned}}$$

$+0.08328125(f\pi^2)^2-0.3125000(f\pi^2)+0.5625000$ 436

$+(0.25000-0.385000f\pi^2)\left\{0.86930118(1+0.1800f\pi^2)-(0.94887221+0.03958959f\pi^2)\right\}$

$-0.1025000(f\pi^2)^2\left\{0.043183105-0.026869720\right\}+$

$+(0.2500-0.38500f\pi^2)^2\left\{8.4410200-15.7229706+7.4410933\right\}$

$+(0.1025000f\pi^2)^2\left\{14.6459494-28.2123232+13.6459506\right\}$

$=0.08328125(f\pi^2)^2-0.3125000(f\pi^2)+0.5625000$

$-(0.25000-0.385000f\pi^2)(0.07957103-0.11688462f\pi^2)-0.001672122(f\pi^2)^2$

$+0.1591427(0.2500-0.38500f\pi^2)^2+0.0008360536(f\pi^2)^2$

$=0.08328125(f\pi^2)^2-0.3125000(f\pi^2)+0.5625000$

$-(0.01989278-0.0598560f\pi^2+0.0450026f\pi^4)-0.001672122(f\pi^2)^2$

$+(0.00994642-0.03063497f\pi^2+0.02358893f\pi^4)+0.0008360536(f\pi^2)^2$

$=0.06103351(f\pi^2)^2-0.28327897(f\pi^2)+0.5525362$

$$6\pi^2 \frac{\sigma}{E}\left(\frac{a}{R}\right)^2 = \left(\frac{a}{R}\right)^4 \left[0.24413404\,\pi^4 f^2 - 0.84983691\,\pi^2 f + 1.1050724\right] \qquad \underline{\underline{477}}$$

$$+ \left(\frac{t}{R}\right)^2 \frac{16\pi^4}{3}$$

$$K = j^2\left[0.040689\,\pi^2 f^2 - 0.1416395 f + \frac{0.1841787}{\pi^2}\right] + \frac{8}{9}\pi^2 \frac{1}{j^2}$$

now

$$\frac{\delta}{t} = \frac{R}{t}\,\frac{1}{2}\left(\frac{a}{R}\right)^2 f, \quad \text{or} \quad f = \frac{2\left(\frac{\delta}{t}\right)}{j^2}$$

$$\therefore \quad K = \pi^2\left[0.162756\left(\frac{\delta}{t}\right)^2 + \frac{8}{9}\right]\frac{1}{j^2} - 0.2832790\left(\frac{\delta}{t}\right) + \frac{0.1841787}{\pi^2} j^2$$

$$\text{Thus} \quad K = 2\left[0.0299762\left(\frac{\delta}{t}\right)^2 + 0.1637144\right]^{\frac{1}{2}} - 0.2832790\left(\frac{\delta}{t}\right)$$

$$\frac{0.0599524\left(\frac{\delta}{t}\right)}{\left[0.0299762\left(\frac{\delta}{t}\right)^2 + 0.1637144\right]^{\frac{1}{2}}} = 0.2832790$$

$$0.00359429\left(\frac{\delta}{t}\right)^2 = 0.00240550\left(\frac{\delta}{t}\right)^2 + 0.01313759$$

$$\left(\frac{\delta}{t}\right)^2_{min} = \frac{0.01313759}{0.00118879} = 11.05123, \qquad \left(\frac{\delta}{t}\right)_{min} = 3.32434$$

$$K_{min} = \left(\frac{0.1199048}{0.2832790} - 0.2832790\right) 3.32434 = 0.4653930$$

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$$\gamma = \pi \cdot \left[0.883685 \frac{f}{t}^{2} + 4.826230 \right]^{\frac{1}{4}}$$

$$\gamma_0 = \pi \sqrt[4]{4.826230} = \pi \sqrt[2]{2.196868} = 1.482183\pi$$

$$\gamma_{min} = \pi \sqrt[4]{14.592036} = \pi \sqrt[2]{3.819952} = 1.954469\pi$$

① $\left(\frac{f}{t}\right)$	② $0.2832790\left(\frac{f}{t}\right)$	③ $0.0299762\left(\frac{f}{t}\right)^2$	④ ③ $+0.1637144$	⑤ ④$^{\frac{1}{2}}$	⑥ K		
0	0	0	0.1637144	0.404617	0.809234		
1	0.2832790	0.0299762	0.1936906	0.440104	0.596929		
2	0.5665580	0.1199048	0.2836191	0.532559	0.498560		
3	0.8498370	0.2697858	0.4335002	0.658407	0.466977		
4	1.1331160	0.4796192	0.6433336	0.802081	0.471046		
5	1.4163950	0.7494050	0.9131194	0.955573	0.494751		
6	1.6996740	1.0791432	1.2428526	1.1148352	0.529996		
7	1.9829530	1.4688338	1.6325482	1.277713	0.572473		
8	2.2662320	1.9184768	2.082191	1.442979	0.619726		
9	2.5495110	2.4280722					
10	2.8327900	2.9976200					
11	3.1160690	3.6271202					

$\lambda = 1.0000$

If the wave length in y- direction is (a/λ) instead of a

$$\nabla^4 F = \frac{E}{R^2}\frac{(f\lambda^2\pi^2)}{2}\Big[-\frac{f\pi^2}{4}\cos\frac{2\pi x}{a} + \Big\{1-\frac{f\pi^2}{4}\Big\}\cos\frac{2\pi\lambda y}{a} + \Big\{1-\frac{f\pi^2}{2}\Big\}\cos\frac{2\pi x}{a}\cos\frac{2\pi\lambda y}{a}$$

$$-\frac{f\pi^2}{4}\Big\{\cos\frac{4\pi x}{a} + \cos\frac{4\pi\lambda y}{a} + \cos\frac{4\pi x}{a}\cos\frac{2\pi\lambda y}{a} + \cos\frac{2\pi x}{a}\cos\frac{4\pi\lambda y}{a}\Big\}\Big]$$

Therefore the particular integral is

$$E\left(\frac{a}{R}\right)^2\frac{f\lambda^2}{8}\Big[-\frac{f\pi^2}{4}\frac{\cos\frac{2\pi x}{a}}{\left(\frac{2\pi}{a}\right)^2} + \frac{1}{\lambda^4}\Big\{1-\frac{f\pi^2}{4}\Big\}\frac{\cos\frac{2\pi\lambda y}{a}}{\left(\frac{2\pi}{a}\right)^2} + \frac{1}{(1+\lambda^2)^2}\Big\{1-\frac{f\pi^2}{2}\Big\}\frac{\cos\frac{2\pi x}{a}\cos\frac{2\pi\lambda y}{a}}{\left(\frac{2\pi}{a}\right)^2}$$

$$-\frac{f\pi^2}{4}\Big\{\frac{1}{4}\frac{\cos\frac{4\pi x}{a}}{\left(\frac{4\pi}{a}\right)^2} + \frac{1}{4\lambda^4}\frac{\cos\frac{4\pi\lambda y}{a}}{\left(\frac{4\pi}{a}\right)^2} + \frac{1}{(4+\lambda^2)^2}\frac{\cos\frac{4\pi x}{a}\cos\frac{2\pi\lambda y}{a}}{\left(\frac{2\pi}{a}\right)^2} + \frac{1}{(1+4\lambda^2)^2}\frac{\cos\frac{2\pi x}{a}\cos\frac{4\pi\lambda y}{a}}{\left(\frac{2\pi}{a}\right)^2}\Big\}\Big]$$

$$F = \frac{E}{\left(\frac{2\pi}{a}\right)^2}\Big[A_2\cosh\left(\frac{2\pi\lambda y}{a}\right) + B_2\left(\frac{2\pi\lambda y}{a}\right)\sinh\left(\frac{2\pi\lambda y}{a}\right)\Big]\cos\frac{2\pi x}{a}$$

$$+\frac{E}{\left(\frac{4\pi}{a}\right)^2}\Big[A_4\cosh\left(\frac{4\pi\lambda y}{a}\right) + B_2\left(\frac{4\pi\lambda y}{a}\right)\sinh\left(\frac{4\pi\lambda y}{a}\right)\Big]\cos\frac{4\pi x}{a} + Ea^2\left(\frac{x}{a}\right)^2 C$$

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$$\frac{\sigma_y}{E} = -\left[A_2 \cosh\left(\frac{2\pi\lambda y}{a}\right) + B_2\left(\frac{2\pi\lambda y}{a}\right)\sinh\frac{2\pi\lambda y}{a}\right]\cos\frac{2\pi x}{a} - \left[A_4 \cosh\left(\frac{4\pi\lambda y}{a}\right) + B_4\left(\frac{4\pi\lambda y}{a}\right)\sinh\left(\frac{4\pi\lambda y}{a}\right)\right]\cos\frac{4\pi x}{a}$$

$$+ 2C + \left(\frac{a}{R}\right)^2 \frac{f\lambda^2}{8}\left[-\frac{f\pi^2}{4}\cos\frac{2\pi x}{a} - \frac{1}{(1+\lambda^2)^2}\left(1-\frac{f\pi^2}{2}\right)\cos\frac{2\pi x}{a}\cos\frac{2\pi\lambda y}{a}\right.$$

$$\left. + \frac{f\pi^2}{4}\left\{\frac{1}{4}\cos\frac{4\pi x}{a} + \frac{4}{(4+\lambda^2)^2}\cos\frac{4\pi x}{a}\cos\frac{2\pi\lambda y}{a} + \frac{1}{(1+4\lambda^2)^2}\cos\frac{2\pi x}{a}\cos\frac{4\pi\lambda y}{a}\right\}\right]$$

$a_2 \cosh 2\pi + 2\pi b_2 \sinh 2\pi = \frac{f\pi^2}{4} + \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{2} - 1\right) + \frac{1}{4(1+4\lambda^2)^2} f\pi^2$
$(a_2 + b_2)\sinh 2\pi + 2\pi b_2 \cosh 2\pi = 0$
$a_4 \cosh 4\pi + 4\pi b_4 \sinh 4\pi = \frac{f\pi^2}{4}\left(\frac{1}{4} + \frac{4}{(4+\lambda^2)^2}\right)$
$(a_4 + b_4)\sinh 4\pi + 4\pi b_4 \cosh 4\pi = 0$

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$$\frac{1}{abE^2}\int_0^a\int_0^b \sigma_y^2\,dx\,dy = \left(\frac{a}{R}\right)^4 \frac{(f\lambda^2)^2}{256}\left[\frac{1}{8}(f\pi^2)^2 + \frac{1}{(1+\lambda^2)^4}\left(\frac{f\pi^2}{2}-1\right)^2 + \frac{(f_1\pi^2)^2}{(4+\lambda^2)^4} + \frac{(f_1\pi^2)^2}{16(1+4\lambda^2)^4} + \frac{(f\pi^2)^2}{128}\right]$$

$$-\left(\frac{a}{R}\right)^4 \frac{(f\lambda^2)^2}{64}\left[\frac{f\pi^2}{4}\left\{\frac{(a_2-b_2)}{2\pi}\sinh 2\pi + b_2\cosh 2\pi\right\} + \frac{1}{(1+\lambda^2)^2}\left(\frac{f_1\pi^2}{2}-1\right)\left\{a_2\frac{\sinh 2\pi}{4\pi} + b_2\frac{\cosh 2\pi}{2}\right\}\right.$$

$$+\frac{f\pi^2}{4(1+4\lambda^2)^2}\left\{\frac{1}{10\pi}a_2\sinh 2\pi + b_2\left(\frac{1}{5}\cosh 2\pi + \frac{3}{50\pi}\sinh 2\pi\right)\right\}$$

$$\left.+\frac{\lambda f\pi^2}{16}\left\{\frac{(a_4-b_4)}{4\pi}\sinh 4\pi + b_4\cosh 4\pi\right\} + \frac{f_1\pi^2}{(1+\lambda^2)^2}\left\{a_4\frac{\sinh 4\pi}{5\pi} + b_4\left(\frac{4}{5}\cosh 4\pi - \frac{3}{25\pi}\sinh 4\pi\right)\right\}\right]$$

$$+\left(\frac{a}{R}\right)^4 \frac{(f\lambda^2)^2}{128}\left[\frac{a_2^2}{2}\left(\frac{\sinh 4\pi}{4\pi}+1\right) + \frac{a_2 b_2}{2}\left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right) + \frac{b_2^2}{2}\left(\frac{16\pi^2+2}{16\pi}\sinh 4\pi - \frac{1}{2}\cosh 4\pi - \frac{4\pi^2}{3}\right)\right.$$

$$\left.+\frac{a_4^2}{2}\left(\frac{\sinh 8\pi}{8\pi}+1\right) + \frac{a_4 b_4}{2}\left(\cosh 8\pi - \frac{\sinh 8\pi}{8\pi}\right) + \frac{b_4^2}{2}\left(\frac{64\pi^2+2}{32\pi}\sinh 8\pi - \frac{1}{2}\cosh 8\pi - \frac{16\pi^2}{3}\right)\right]$$

$$= A_1$$

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$$\frac{1}{abE^2}\int_0^a\int_0^b \sigma_x^2\,dx\,dy = \lambda^4\left(\frac{a}{R}\right)^4\frac{(f\lambda^2)^2}{256}\Big[\frac{2}{\lambda^8}\left(\frac{f\pi^2}{4}-1\right)^2+\frac{1}{(1+\lambda^2)^4}\left(\frac{f\pi^2}{2}-1\right)^2+\frac{2}{\lambda^8}\left(\frac{f\pi^2}{16}\right)^2+\frac{(f\pi^2)^2}{16(4+\lambda^2)^4}$$

$$+\frac{(f\pi^2)^2}{(1+4\lambda^2)^4}\Big]$$

$$+\lambda^4\left(\frac{a}{R}\right)^4\frac{(f\lambda^2)^2}{64}\Big[\frac{1}{\lambda^4}\left(\frac{f\pi^2}{4}-1\right)\left\{\frac{(a_2+b_2)}{2\pi}\sinh 2\pi+b_2\cosh 2\pi\right\}+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{2}-1\right)\left\{(a_2+2b_2)\frac{\sinh 2\pi}{4\pi}+b_2\frac{\cosh 2\pi}{2}\right\}$$

$$+\frac{f\pi^2}{(1+4\lambda^2)^2}\left\{\frac{1}{10\pi}(a_2+2b_2)\sinh 2\pi+b_2\left(\frac{1}{5}\cosh 2\pi+\frac{3}{50\pi}\sinh 2\pi\right)\right\}$$

$$+\frac{f\pi^2}{16\lambda^4}\left\{\frac{(a_4+b_4)}{4\pi}\sinh 4\pi+b_4\cosh 4\pi\right\}+\frac{f\pi^2}{4(4+\lambda^2)^2}\left\{\frac{1}{5\pi}(a_4+2b_4)\sinh 4\pi+b_4\left(\frac{4}{5}\cosh 4\pi-\frac{3}{25\pi}\sinh 4\pi\right)\right\}\Big]$$

$$+\left(\frac{a}{R}\right)^4\lambda^4\frac{(f\lambda^2)^2}{128}\Big[\frac{(a_2+2b_2)^2}{2}\left(\frac{\sinh 4\pi}{4\pi}+1\right)+\frac{(a_2+2b_2)b_2}{2}\left(\cosh 4\pi-\frac{\sinh 4\pi}{4\pi}\right)+\frac{b_2^2}{2}\left(\frac{16\pi^2+2}{16\pi}\sinh 4\pi-\frac{1}{2}\cosh 4\pi-\frac{4\pi^2}{3}\right)$$

$$+\frac{(a_4+2b_4)^2}{2}\left(\frac{\sinh 8\pi}{8\pi}+1\right)+\frac{(a_4+2b_4)b_4}{2}\left(\cosh 8\pi-\frac{\sinh 8\pi}{8\pi}\right)+\frac{b_4^2}{2}\left(\frac{64\pi^2+2}{32\pi}\sinh 8\pi-\frac{1}{2}\cosh 8\pi-\frac{16\pi^2}{3}\right)\Big]$$

$$= A_2$$

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$$\frac{1}{abE^2}\int_0^a\int_0^b \tau_{xy}^2\,dx\,dy = \lambda^2\left(\frac{a}{R}\right)^4 \frac{(1-\nu)^2}{256}\left[\frac{1}{(1+\lambda^2)^4}\left(\frac{(1-\nu)\pi^2}{2}-1\right)^2 + \frac{\left((1-\nu)\pi^2\right)^2}{4(4+\lambda^2)^4} + \frac{\left((1-\nu)\pi^2\right)^2}{4(1+4\lambda^2)^4}\right]$$

$$+\lambda^2\left(\frac{a}{R}\right)^4 \frac{(1-\nu)^2}{64}\left[\frac{1}{(1+\lambda^2)^2}\left(\frac{(1-\nu)\pi^2}{2}-1\right)\left\{-(a_2+b_2)\frac{\sinh 2\pi}{4\pi} + b_2\left(\frac{\sinh 2\pi}{4\pi}-\frac{1}{2}\cosh 2\pi\right)\right\}\right.$$

$$+\frac{(1-\nu)\pi^2}{2(1+4\lambda^2)^2}\left\{-(a_2+b_2)\frac{\sinh 2\pi}{5\pi} + b_2\left(-\frac{2}{5}\cosh 2\pi + \frac{2}{25\pi}\sinh 2\pi\right)\right\}$$

$$\left.+\frac{(1-\nu)\pi^2}{2(4+\lambda^2)^2}\left\{-(a_4+b_4)\frac{\sinh 4\pi}{10\pi} + b_4\left(-\frac{2}{5}\cosh 4\pi + \frac{4}{25\pi}\sinh 4\pi\right)\right\}\right]$$

$$+\lambda^2\left(\frac{a}{R}\right)^4 \frac{(1-\nu)^2}{128}\left[\frac{(a_2+b_2)^2}{2}\left\{\frac{\sinh 4\pi}{4\pi}-1\right\} + \frac{b_2(a_2+b_2)}{2}\left\{\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right\} + \frac{b_2^2}{2}\left\{\frac{4\pi^2}{3} + \frac{16\pi^2+2}{16\pi}\sinh 4\pi - \frac{\cosh 4\pi}{2}\right\}\right.$$

$$\left.+\frac{(a_4+b_4)^2}{2}\left\{\frac{\sinh 8\pi}{8\pi}-1\right\} + \frac{b_4(a_4+b_4)}{2}\left\{\cosh 8\pi - \frac{\sinh 8\pi}{8\pi}\right\} + \frac{b_4^2}{2}\left\{\frac{16\pi^2}{3} + \frac{64\pi^2+2}{32\pi}\sinh 8\pi - \frac{\cosh 8\pi}{2}\right\}\right]$$

A_3

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$$\frac{1}{a}\int_0^a \sin\frac{2\pi x}{a}\sinh(\sqrt{2}\lambda)\frac{\pi x}{a}\,dx = \frac{1}{2\pi i}\int_0^a \left[\cosh\frac{\pi x}{a}(\sqrt{2}\lambda+2i) - \cosh\frac{\pi x}{a}(\sqrt{2}\lambda-2i)\right] d\left(\frac{\pi x}{a}\right)$$

$$= \frac{1}{2\pi i}\left[\frac{1}{\sqrt{2}\lambda+2i}\sinh\pi(\sqrt{2}\lambda+2i) - \frac{1}{\sqrt{2}\lambda-2i}\sinh\pi(\sqrt{2}\lambda-2i)\right] = -\frac{2}{\pi}\,\frac{\sinh\pi\sqrt{2}\lambda}{4+2\lambda^2} = -\frac{1}{\pi}\,\frac{\sinh\pi\sqrt{2}\lambda}{2+\lambda^2}$$

$$\frac{1}{a}\int_0^a \frac{\pi x}{a}\sin\frac{2\pi x}{a}\,dx = \frac{1}{4\pi}\Big[\sin\theta - \theta\cos\theta\Big]_0^{2\pi} = -\frac{1}{2}$$

$$\frac{1}{a}\int_0^a \frac{\pi x}{a}\sin\frac{2\pi x}{a}\cos\frac{\pi x}{a}\,dx = \frac{1}{2\pi}\int_0^a \frac{\pi x}{a}\left[\sin\frac{3\pi x}{a} + \sin\frac{\pi x}{a}\right] d\left(\frac{\pi x}{a}\right)$$

$$= \frac{1}{2\pi}\left[\frac{1}{9}\left(\sin\theta - \theta\cos\theta\right)_0^{3\pi} + \left(\sin\theta - \theta\sin\theta\right)_0^{\pi}\right] = \frac{1}{2}\left[\frac{1}{3}+1\right] = \frac{2}{3}$$

$$\frac{1}{a}\int_0^a \sinh^2\frac{\sqrt{2}\lambda\pi x}{a}\,dx = \frac{1}{2\pi}\int_0^{\pi}\left(\cosh 2\sqrt{2}\lambda\theta - 1\right)d\theta = \frac{1}{2\pi}\left[\frac{\sinh 2\sqrt{2}\lambda\pi}{2\sqrt{2}\lambda} - \pi\right] = \frac{\sinh 2\sqrt{2}\lambda\pi}{4\sqrt{2}\lambda\pi} - \frac{1}{2}$$

$$\frac{1}{a}\int_0^a \frac{\pi x}{a}\sinh(\sqrt{2}\lambda)\frac{\pi x}{a}\,dx = \frac{1}{2\lambda^2\pi}\left[\cdot\sqrt{2}\lambda\pi\cosh\sqrt{2}\lambda\pi - \sinh\sqrt{2}\lambda\pi\right]$$

$$\frac{1}{a}\int_0^a \left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\sinh\sqrt{2}\lambda\left(\frac{\pi x}{a}\right)dx = \frac{1}{2\pi}\int_0^{\pi}\theta\left[\sinh\theta(\sqrt{2}\lambda+i) + \sinh\theta(\sqrt{2}\lambda-i)\right]d\theta$$

$$= \frac{1}{2\pi}\left[\frac{1}{(\sqrt{2}\lambda+i)^2}\left\{(\sqrt{2}\lambda+i)\pi\cosh\pi(\sqrt{2}\lambda+i) - \sinh\pi(\sqrt{2}\lambda+i)\right\} + \frac{1}{(\sqrt{2}\lambda-i)^2}\left\{(\sqrt{2}\lambda-i)\pi\cosh\pi(\sqrt{2}\lambda-i) - \sinh\pi(\sqrt{2}\lambda-i)\right\}\right]$$

$$= -\frac{\sqrt{2}\lambda\cosh\sqrt{2}\lambda\pi}{1+2\lambda^2} + \frac{1}{\pi}\,\frac{(2\lambda^2-1)\sinh\sqrt{2}\lambda\pi}{(1+2\lambda^2)^2}$$

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$$\frac{1}{a}\int_0\left(\frac{\pi x}{a}\right)^2 dx = \pi^2\int_0^1 \theta^2 d\theta = \frac{\pi^2}{3}$$

$$\frac{1}{a}\int_0^a\left(\frac{\pi x}{a}\right)^2\cos\frac{\pi x}{a}\,dx = \frac{1}{\pi}\int_0^\pi \theta^2\cos\theta\, d\theta = \frac{1}{\pi}\left[-2\pi\right] = -2$$

$$\frac{1}{a}\int_0^a\left(\frac{\pi x}{a}\right)^2\cos^2\frac{\pi x}{a}\,dx = \frac{1}{\pi}\int_0^\pi \theta^2\cos^2\theta\, d\theta = \frac{1}{2\pi}\int_0^\pi\theta^2(1+\cos 2\theta)\,d\theta = \frac{1}{2\pi}\left[\frac{\pi^3}{3}+\frac{1}{8}(4\pi)\right] = \frac{\pi^2}{6}+\frac{1}{4}$$

$$\frac{1}{E^2ab}\int_0^a\int_0^b \tau_{xy}^2\,dx\,dy = \left(\frac{a}{R}\right)^4\left(\frac{f\lambda}{8}\right)^2\Bigg[\frac{1}{4}\left\{\frac{f\pi^2}{8}\frac{\lambda^2}{(1+2\lambda^2)}+\frac{1+\lambda^2}{(1+2\lambda^2)^2}\right\}^2+\frac{1}{4}\left(\frac{f\pi^2}{32}\right)^2\left(\frac{\lambda^2}{2+\lambda^2}\right)^2+\frac{1}{4}\left(\frac{f\pi^2}{16}\right)^2\left(\frac{3\lambda^2}{1+8\lambda^2}\right)^2$$

$$+\frac{1}{4}\left(\frac{f\pi^2}{128}\right)^2\left(\frac{3\lambda^2}{1+2\lambda^2}\right)^2 + \cancel{\frac{1}{4}\left(\frac{f\pi^2}{16}\right)^2\left(\frac{3\lambda^2}{1+8\lambda^2}\right)^2} + \cancel{\frac{1}{4}\left(\frac{f\pi^2}{128}\right)^2\left(\frac{3\lambda^2}{1+2\lambda^2}\right)^2}$$

$$+\left\{\frac{f\pi^2}{8}\frac{\lambda^2}{(1+2\lambda^2)}+\frac{1+\lambda^2}{(1+2\lambda^2)^2}\right\}\left\{\frac{1}{2}\frac{1}{(1+2\lambda^2)^2}-\frac{1}{2}+\frac{1}{4}\frac{\lambda^2}{(1+2\lambda^2)}\right\}$$

$$+\frac{f\pi^2}{32}\frac{\lambda^2}{(2+\lambda^2)}\left\{-\frac{1}{2}\frac{1}{(1+2\lambda^2)(2+\lambda^2)}+\frac{1}{4}-\frac{2}{3}\frac{\lambda^2}{(1+2\lambda^2)}\right\}$$

$$+\frac{1}{2}\left\{\frac{\left(\frac{\sinh 2\sqrt{2}\lambda\pi}{4\sqrt{2}\lambda\pi}-\frac{1}{2}\right)\pi^2}{4(1+2\lambda^2)^2\sinh^2(\sqrt{2}\lambda\pi)}-\frac{1}{2}\frac{\pi}{(1+2\lambda^2)\sinh(\sqrt{2}\lambda\pi)}\frac{1}{2\lambda^2\pi}\left[\sqrt{2}\lambda\pi\cosh\sqrt{2}\lambda\pi-\sinh\sqrt{2}\lambda\pi\right]\right.$$

$$-\frac{\lambda^2}{(1+2\lambda^2)}\frac{\pi}{(1+2\lambda^2)\sinh(\sqrt{2}\lambda\pi)}\left[-\frac{\sqrt{2}\lambda\cosh\sqrt{2}\lambda\pi}{1+2\lambda^2}+\frac{1}{\pi}\frac{(2\lambda^2-1)\sinh\sqrt{2}\lambda\pi}{(1+2\lambda^2)^2}\right]+\frac{\pi^2}{12}$$

$$\left.-\frac{2\lambda^2}{(1+2\lambda^2)}+\frac{\lambda^4}{(1+2\lambda^2)^2}\left(\frac{\pi^2}{6}+\frac{1}{4}\right)\right\}\Bigg]$$

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$$\frac{E_1}{2Eabt} = \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\left[H_1(\lambda)(1/\pi^2)^2 + H_2(\lambda)(1/\pi^2) + H_3(\lambda)\right] + \left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right)\lambda^2\frac{\sigma}{E}\left\{\frac{1}{2\lambda^2} - \frac{3}{128}(1/\pi^2)\right\}$$

$$H_1(\lambda) = \frac{105}{32768} + \frac{9\lambda^4}{32768} + \frac{77\lambda^4}{65536} + \frac{(1-4\lambda^2)^2\lambda^4}{2048(1+8\lambda^2)^2} + \frac{(1-\lambda^2)^2\lambda^4}{32768(1+2\lambda^2)^2} + \frac{\lambda^4}{512(1+2\lambda^2)^2}$$

$$+ \frac{\lambda^4}{2048(2+\lambda^2)^2} + \frac{\lambda^6}{256(1+2\lambda^2)^2} + \frac{\lambda^6}{4096(2+\lambda^2)^2} + \frac{9\lambda^6}{1024(1+8\lambda^2)^2} + \text{———}$$

$$+ \frac{9\lambda^6}{65536(1+2\lambda^2)^2} + \cancel{\frac{9\lambda^6}{1024(1+8\lambda^2)^2}} + \cancel{\frac{9\lambda^6}{65536(1+2\lambda^2)^2}}$$

$$\boxed{H_1(\lambda) = \frac{105}{32768} + \frac{95\lambda^4}{65536} + \frac{\lambda^4(1+2\lambda^2)}{2048(1+8\lambda^2)} + \frac{\lambda^4(130+\lambda^2)}{65536(1+2\lambda^2)} + \frac{3\lambda^6}{4096(2+\lambda^2)^2}}$$

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$$H_2(\lambda) = -\frac{5}{48} - \frac{3}{256}\lambda^2 - \frac{\lambda^6}{32(1+2\lambda^2)^3} - \frac{\lambda^4}{64(1+2\lambda^2)^2} - \frac{\lambda^4}{16(1+2\lambda^2)^3} - \frac{\lambda^4}{96(2+\lambda^2)(1+2\lambda^2)} + \cancel{\frac{\lambda^4}{64(1+2\lambda^2)(2+\lambda^2)^2}}$$

$$+ \frac{\lambda^4(1+\lambda^2)}{16(1+2\lambda^2)^3} + \frac{\lambda^6}{16(1+2\lambda^2)^3} - \frac{\lambda^4}{16(1+2\lambda^2)} + \frac{\lambda^6}{32(1+2\lambda^2)^2} - \cancel{\frac{\lambda^4}{64(1+2\lambda^2)(2+\lambda^2)^2}} + \frac{\lambda^4}{128(2+\lambda^2)}$$

$$- \frac{\lambda^6}{48(1+2\lambda^2)(2+\lambda^2)}$$

$$\boxed{H_2(\lambda) = -\frac{5}{48} - \frac{3\lambda^2}{256} - \frac{3\lambda^4}{64(1+2\lambda^2)} - \frac{\lambda^6}{384(2+\lambda^2)}}$$

$$H_3(\lambda) = \frac{\pi^2}{8} - \cancel{\frac{3}{16}} + \cancel{\frac{1}{8}} + \frac{\lambda^4}{8(1+2\lambda^2)^2} + \cancel{\frac{1}{16}} + \frac{\lambda^4}{8(1+2\lambda^2)^3} + \frac{\lambda^4}{2(1+2\lambda^2)^4} - \frac{\lambda^2}{4(1+2\lambda^2)} - \cancel{\frac{1}{8(1+2\lambda^2)}}$$

$$+ \frac{1}{4}\frac{\lambda^4}{(1+2\lambda^2)^2}\left(\cancel{\frac{\pi^2}{6}} - \frac{1}{4}\right) + \frac{\lambda^2}{4(1+2\lambda^2)^3}(\sqrt{2}\lambda\pi)\coth\sqrt{2}\lambda\pi - \frac{\lambda^4}{(1+2\lambda^2)^4} + \frac{\pi^2\lambda^2\left(\frac{1}{2} + \frac{1}{4\sqrt{2}\pi\lambda}\sinh 2\sqrt{2}\lambda\pi\right)}{4(1+2\lambda^2)^2(\cosh 2\sqrt{2}\lambda\pi - 1)}$$

$$+ \frac{\lambda^2(1+\lambda^2)^2}{4(1+2\lambda^2)^4} + \frac{\lambda^2(1+\lambda^2)}{2(1+2\lambda^2)^4} - \frac{\lambda^2(1+\lambda^2)}{2(1+2\lambda^2)^2} + \frac{\lambda^4(1+\lambda^2)}{4(1+2\lambda^2)^3} + \frac{\pi^2\lambda^2\left(-\frac{1}{2} + \frac{1}{4\sqrt{2}\pi\lambda}\sinh 2\sqrt{2}\lambda\pi\right)}{4(1+2\lambda^2)^2(\cosh 2\sqrt{2}\lambda\pi - 1)}$$

$$- \frac{1}{8(1+2\lambda^2)}(\sqrt{2}\lambda\pi)\coth\sqrt{2}\lambda\pi + \cancel{\frac{1}{8}\frac{1}{(1+2\lambda^2)}} + \frac{\lambda^4}{2(1+2\lambda^2)^3}\sqrt{2}\pi\lambda\coth\sqrt{2}\lambda\pi + \frac{(1-2\lambda^2)\lambda^4}{2(1+2\lambda^2)^4} + \cancel{\frac{\pi^2}{24}}\lambda^2$$

$$- \frac{\lambda^4}{(1+2\lambda^2)} + \frac{\lambda^6}{2(1+2\lambda^2)^2}\left(\cancel{\frac{\pi^2}{6}} + \frac{1}{4}\right)$$

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$$H_3(\lambda) = \frac{\pi^2}{8} + \frac{3\lambda^2(2-\lambda^2)}{8(1+2\lambda^2)^3} + \frac{\lambda^4(3+2\lambda^2)}{8(1+2\lambda^2)^3} - \frac{\lambda^2(1+4\lambda^2)}{4(1+2\lambda^2)} - \frac{\lambda^2(8+9\lambda^2-2\lambda^4)}{16(1+2\lambda^2)^2}$$

$$+ \frac{\pi^2}{24}\lambda^2\left[\frac{\lambda^2}{(1+2\lambda^2)} + 1\right] + \left[\frac{\lambda^2}{4(1+2\lambda^2)^2} - \frac{1}{8(1+2\lambda^2)} + \frac{1}{16(1+2\lambda^2)^2}\right]\sqrt{2}\pi\lambda \coth\sqrt{2}\lambda\pi$$

$$\boxed{H_3(\lambda) = \frac{\pi^2}{8} + \frac{\lambda^2(3+\lambda^4)}{4(1+2\lambda^2)^3} - \frac{3\lambda^2(4+11\lambda^2+10\lambda^4)}{16(1+2\lambda^2)^2} + \frac{\pi^2}{24}\lambda^2\left[\frac{\lambda^2}{(1+2\lambda^2)} + 1\right] - \frac{1}{16(1+2\lambda^2)^2}\sqrt{2}\pi\lambda \coth\sqrt{2}\lambda\pi}$$

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$$\frac{E_2}{E a b t} = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{8}\right)^2\left[\frac{3}{8}+\frac{3}{8}\lambda^4+\frac{1}{4}\lambda^2\right] = \left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{8}\right)^2\left[\frac{1}{32}(1+\lambda^4)+\frac{1}{48}\lambda^2\right]$$

Thus total energy expression

$$= \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\left[H_1(\lambda)(f\pi^2)^2 + H_2(\lambda)(f\pi^2) + H_3(\lambda)\right] + \left(\frac{t}{R}\right)^2\left(\frac{f}{8}\right)^2\left[\frac{1}{32}(1+\lambda^4)+\frac{1}{48}\lambda^2\right]\pi^4$$

$$-\left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right)^2\pi^2\left(\frac{3\lambda^2}{8}\right)\frac{\sigma}{E} \quad - \frac{1}{3}\left(\frac{\sigma}{F}\right)^2$$

$$\therefore \quad \frac{\sigma}{E}\,\frac{3\pi^2\lambda^2}{4} = \left(\frac{a}{R}\right)^2\left[4H_1 f^2\pi^4 + 3H_2 f\pi^2 + 2H_3\right] + \frac{\left(\frac{t}{R}\right)^2}{\left(\frac{a}{R}\right)^2}\left[\frac{1}{16}(1+\lambda^4)+\frac{1}{24}\lambda^2\right]\pi^4$$

$$\lambda^2 K = \gamma^2\left[\frac{16}{3}H_1 f^2\pi^2 + 4H_2 f + \frac{8H_3}{3\pi^2}\right] + \frac{\pi^2}{\gamma^2}\left[\frac{1}{12}(1+\lambda^4)+\frac{1}{18}\lambda^2\right]$$

$$= \frac{\pi^2}{\gamma^2}\left[\frac{64}{3}H_1\left(\frac{f}{t}\right)^2 + \left\{\frac{1}{12}(1+\lambda^4)+\frac{1}{18}\lambda^2\right\}\right] + \frac{8H_3}{3}\frac{\gamma^2}{\pi^2} + 8H_2\left(\frac{f}{t}\right)$$

$$= 2\left[\frac{512}{9}H_1H_3\left(\frac{f}{t}\right)^2 + H_3\left\{\frac{2}{9}(1+\lambda^4)+\frac{4}{27}\lambda^2\right\}\right]^{\frac{1}{2}} + 8H_2\left(\frac{f}{t}\right)$$

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X

$$-a_2\,\frac{\sinh 2\pi}{2\pi}\left\{\left[\frac{1}{4(1+\lambda^2)}+\frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2}+\frac{1}{20(1+4\lambda^2)}+\frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)^2}\right](4\pi^2)+\left[1-\frac{1}{2(1+\lambda^2)}-\frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2}\right]\right\}$$

$$+b_2\left[\frac{\sinh 2\pi}{2\pi}\left\{\left[\frac{1}{2}+\frac{\lambda^4}{2(1+\lambda^2)^2}+\frac{2}{5}\,\frac{\lambda^2(\lambda^2-1)}{(1+4\lambda^2)^2}+\frac{(2\lambda^2+3)(6\lambda^2-1)}{100(1+4\lambda^2)^2}\right](4\pi^2)-\left[1+\frac{\lambda^4}{(1+\lambda^2)^2}\right]\right\}\right.$$

$$\left.-\cosh 2\pi\left\{\left[\frac{1}{4(1+\lambda^2)}+\frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2}+\frac{1}{20(1+4\lambda^2)}+\frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)^2}\right](4\pi^2)+\left[1-\frac{1}{2(1+\lambda^2)}-\frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2}\right]\right\}\right]$$

$$-a_4\,\frac{\sinh 4\pi}{4\pi}(4\pi^2)\left\{\frac{1}{5(4+\lambda^2)}+\frac{\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2}\right\}$$

$$+b_4(4\pi^2)\left[\frac{\sinh 4\pi}{4\pi}\left\{\frac{1}{8}+\frac{(2+3\lambda^2)(6-\lambda^2)}{25(4+\lambda^2)^2}-\frac{2\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2}\right\}-\cosh 4\pi\left\{\frac{1}{5(4+\lambda^2)}+\frac{\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2}\right\}\right]$$

$$\cosh 2\pi\, a_2+2\pi\sinh 2\pi\, b_2=(4\pi^2)\left[\frac{1}{4}+\frac{1}{2(1+\lambda^2)^2}+\frac{1}{4(1+4\lambda^2)^2}\right]-\frac{1}{(1+\lambda^2)^2}$$

$$\sinh 2\pi\, a_2+(\sinh 2\pi+2\pi\cosh 2\pi)\, b_2=0$$

$$\frac{1}{2}(\sinh 4\pi+4\pi)b_2=-\sinh 2\pi\Big[\qquad\qquad\Big]$$

$$\boxed{b_2=-\frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi}+1}\left[(4\pi^2)\left\{\frac{1}{4}+\frac{1}{2(1+\lambda^2)^2}+\frac{1}{4(1+4\lambda^2)^2}\right\}-\frac{1}{(1+\lambda^2)^2}\right]}$$

$$a_2=+\frac{\frac{\sinh 2\pi}{2\pi}+\cosh 2\pi}{\frac{\sinh 4\pi}{4\pi}+1}\Big[\qquad\qquad\Big]$$

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$$a_4 = + \frac{\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi}{\frac{\sinh 8\pi}{8\pi} + 1} \quad f\,\pi^2 \left\{ \frac{1}{16} + \frac{1}{(4+\lambda^2)^2} \right\}$$

$$b_4 = - \frac{\frac{\sinh 4\pi}{4\pi}}{\frac{\sinh 8\pi}{8\pi} + 1} \quad f\,\pi^2 \left\{ \frac{1}{16} + \frac{1}{(4+\lambda^2)^2} \right\}$$

$$\frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi} + 1} = \frac{42.613218}{11410.473} = 0.0037345707 \ ; \quad \frac{\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi}{\frac{\sinh 4\pi}{4\pi} + 1} = 0.027199735$$

$$\frac{\frac{\sinh 4\pi}{4\pi}}{\frac{\sinh 8\pi}{8\pi} + 1} = 0.00000677468 55 , \quad \frac{\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi}{\frac{\sinh 8\pi}{8\pi} + 1} = 0.0000946 2 1164$$

Let us put

$$g_1(\lambda) = \frac{1}{4(1+\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2} + \frac{1}{20(1+4\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)^2}$$

$$g_2(\lambda) = 1 - \frac{1}{2(1+\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2}$$

$$g_3(\lambda) = \frac{1}{2} + \frac{\lambda^4}{2(1+\lambda^2)^2} + \frac{2}{5}\,\frac{\lambda^2(\lambda^2-1)}{(1+4\lambda^2)^2} + \frac{(2\lambda^2+3)(6\lambda^2-1)}{100(1+4\lambda^2)^2}$$

$$g_4(\lambda) = 1 + \frac{\lambda^4}{(1+\lambda^2)^2}$$

$$g_5(\lambda) = \frac{1}{5(4+\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2}$$

$$g_6(\lambda) = \frac{1}{8} + \frac{(2+3\lambda^2)(6-\lambda^2)}{25(4+\lambda^2)^2} - \frac{2\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2}$$

$$H_1(\lambda) = \frac{1}{4} + \frac{1}{2(1+\lambda^2)^2} + \frac{1}{4(1+4\lambda^2)^2}$$

$$H_2(\lambda) = \frac{1}{(1+\lambda^2)^2}$$

$$H_3(\lambda) = \frac{1}{16} + \frac{1}{(4+\lambda^2)^2}$$

$$-\left[H_1(f\pi^2)-H_2\right]\left[\frac{\frac{\sinh 2\pi}{2\pi}+\cosh 2\pi}{\frac{\sinh 4\pi}{4\pi}+1}\cdot\frac{\sinh 2\pi}{2\pi}\left(g_1 f\pi^2+g_2\right)+\frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi}+1}\frac{\sinh 2\pi}{2\pi}\left(g_3 f\pi^2-g_4\right)\right.$$

$$\left.-\frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi}+1}\cosh 2\pi\left(g_1 f\pi^2+g_2\right)\right]$$

$$-H_3(f\pi^2)^2\left[\frac{\frac{\sinh 4\pi}{4\pi}+\cosh 4\pi}{\frac{\sinh 8\pi}{8\pi}+1}\cdot\frac{\sinh 4\pi}{4\pi}g_5+\frac{\frac{\sinh 4\pi}{4\pi}}{\frac{\sinh 8\pi}{8\pi}+1}\frac{\sinh 4\pi}{4\pi}g_6-\frac{\frac{\sinh 4\pi}{4\pi}\cdot\cosh 4\pi}{\frac{\sinh 8\pi}{8\pi}+1}g_5\right]$$

$$=-\underbrace{\frac{\left(\frac{\sinh 2\pi}{2\pi}\right)^2}{\frac{\sinh 4\pi}{4\pi}+1}}_{0.15914286}\left[H_1(f\pi^2)-H_2\right]\left[\underbrace{(g_1+g_3)}_{\frac{77}{100}}(f\pi^2)-\underbrace{(g_4-g_2)}_{\frac{1}{2}}\right]$$

0.12254000 $f\pi^2$)

− 0.07957143

$$-\underbrace{\frac{\left(\frac{\sinh 4\pi}{4\pi}\right)^2}{\frac{\sinh 8\pi}{8\pi}+1}}_{0.07957486}H_3(f\pi^2)^2\left[\underbrace{g_5+g_6}_{\frac{41}{200}}\right]=F_1$$

0.01631338⁴

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$$F_2 = \left[H_1(\theta\pi^2) - H_2\right]^2\Big[\left\{(1+\lambda^2)^2\, 2.11025500 + 0.00018495(1-\lambda^2)^2\right\} - \left\{0.28974167(3\lambda^4+2\lambda^2-1)\right.$$

$$\left. + 3.64100103\,(1+\lambda^2)^2 - 0.00010158\,\lambda^2(1-\lambda^2)\right\} + \Big\{0.03978203\left[4\pi^2(1+\lambda^2)^2 + \tfrac{1}{2}(5\lambda^4+2\lambda^2+1)\right] + 0.24995787(3\lambda^4+2\lambda^2-1)$$

$$+ 0.00003488\left[2(2\lambda^4-1) - \tfrac{4\pi^2}{3}(1-\lambda^2)^2\right]\Big\}\Big]$$

$$+ \left[H_3(\theta\pi^2)\right]^2\Big[3.66148735\,(1+\lambda^2)^2 - \left\{0.26989437(3\lambda^4+2\lambda^2-1) + 6.78318640(1+\lambda^2)^2\right\}$$

$$+ \left\{0.01989437\left[16\pi^2(1+\lambda^2)^2 + \tfrac{1}{9}(5\lambda^4+2\lambda^2+1)\right] + 0.2500000\,(3\lambda^4+2\lambda^2-1)\right\}\Big]$$

$$F_3 = \left\{\frac{17}{512}\left(1+\frac{1}{\lambda^4}\right) + \frac{1}{16(1+\lambda^2)^2} + \frac{1}{64(4+\lambda^2)^2} + \frac{1}{64(1+4\lambda^2)^2}\right\}(\theta\pi^2)^2$$

$$- \left\{\frac{1}{4\lambda^4} + \frac{1}{4(1+\lambda^2)^2}\right\} + \left\{\frac{1}{2\lambda^4} + \frac{1}{4(1+\lambda^2)^2}\right\}$$

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$$\lambda = 1.000$$

$$F_3 = (0.06640625 + 0.015625 + 0.00125)(f\pi^2)^2 - (0.25 + 0.0625)(f\pi^2) + (0.5 + 0.0625)$$

$$= 0.08328125 (f\pi^2)^2 - 0.3125 (f\pi^2) + 0.5625$$

$$H_1(\lambda) = 0.25 + 0.125 + 0.01 = 0.385, \qquad H_2(\lambda) = 0.25.$$

$$H_3(\lambda) = 0.0625 + 0.04 = 0.1025$$

$$F_1 = -\left[0.385(f\pi^2) - 0.25\right]\left[0.12254000(f\pi^2) - 0.07957143\right] - 0.016313364 \times 0.1025 (f\pi^2)^2$$

$$= -0.04885002 (f\pi^2)^2 + 0.06127000 (f\pi^2) - 0.01989286$$

$$F_2 = \left[0.385(f\pi^2) - 0.25\right]^2 0.15920493 + (0.1025 f\pi^2)^2 0.07957575$$

$$= 0.02359815 (f\pi^2)^2 - 0.03064695 (f\pi^2) + 0.00995031 + 0.00083604 (f\pi^2)^2$$

$$6\pi^2 \frac{\sigma}{E} = \left(\frac{a}{R}\right)^2 \left\{ 0.23546168\, \pi^4 f^2 - 0.84563085\, \pi^2 f + 0.552\underset{\times 2}{5}5745 \right\} + \frac{\left(\frac{t}{R}\right)^2}{\left(\frac{a}{R}\right)^2} \pi^2 \frac{16}{3}$$

$$K = \gamma^2 \left\{ 0.03924361\, \pi^2 f^2 - 0.14093848\, f + \frac{0.0920929\overset{\times 2}{1}}{\pi^2} \right\} + \frac{1}{\gamma^2} \pi^2\, 0.888889$$

$$= \frac{\pi^2}{\gamma^2} \left\{ 0.15697444 \left(\frac{\delta}{t}\right)^2 + 0.8888889 \right\} + \frac{0.09\overset{\times 2}{2}09291\, \gamma^2}{\pi^2} - 0.28187696 \left(\frac{\delta}{t}\right)$$

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$$K = 2\left\{ 0.02891247\left(\frac{\delta}{t}\right)^2 + 0.16372073 \right\}^{\frac{1}{2}} - 0.28187696\left(\frac{\delta}{t}\right)$$

$$K_0 = 0.8092$$

$$0.05782494\left(\frac{\delta}{t}\right) = 0.28187696\left\{ 0.02891247\left(\frac{\delta}{t}\right)^2 + 0.16372073 \right\}^{\frac{1}{2}}$$

$$0.00334372\left(\frac{\delta}{t}\right)^2 = 0.01300837,$$
$$-0.00229723$$

$$\left(\frac{\delta}{t}\right)^2 = \frac{0.01300837}{0.00104649} = 12.4305$$

$$\left(\frac{\delta}{t}\right) = 3.5257$$

$$K_{min} = 0.4527$$

①	②	③	④	⑤			
(δ/t)	$0.0289124\left(\frac{\delta}{t}\right)^2$	② +0.16372	2 ③$^{\frac{1}{2}}$	K			
1	0.028912	0.19263	0.8772	0.5953			
2	0.115648	0.27937	1.0572	0.4934			
3	0.260208	0.42393	1.3020	0.4564			
4	0.462592	0.6[illegible]31	1.5828	0.4553			
5	[illegible]	[illegible]	1.8832	0.4738			
6	1.040832	1.20455	2.1975	0.5062			
7	1.416688	1.58041	2.5142	0.5411			
8	1.850368	2.01409	2.8384	0.5834			
9	2.341872	2.50559	3.1658	0.6289			
10	2.89120	3.05492	3.4956	0.6768			

$$\lambda = 1.200$$

$$F_3 = \left\{0.05626085 + 0.01049785 + 0.000527986 + 0.00034192\right\}(f\pi^2)^2$$

$$- \left\{0.12056327 + 0.04199140\right\}(f\pi^2) + \left\{0.24112654 + 0.04199140\right\}$$

$$= 0.06762861\,(f\pi^2)^2 - 0.16255467\,(f\pi^2) + 0.28311794$$

$$H_1(\lambda) = 0.25 + 0.08398280 + 0.00547075 = 0.33945355$$

$$H_2(\lambda) = 0.16796560$$

$$H_3(\lambda) = 0.0625 + 0.03379109 = 0.09629109$$

$$F_1 = -\left[0.33945355\,(f\pi^2) - 0.16796560\right]\left[0.12254000\,(f\pi^2) - 0.07957143\right] - 0.00157083\,(f\pi^2)^2$$

$$= -0.04316747\,(f\pi^2)^2 + 0.04759331\,(f\pi^2) - 0.01336526$$

$$F_2 = \left[0.33945355\,(f\pi^2) - 0.16796560\right]^2\left[0.19609603\right] + 0.0990[illegible]9\left[0.09629109\right]^2[illegible]$$

$$= 0.1[illegible]\left[0.1152[illegible]\,(f\pi^2)^2 - 0.11[illegible]\,(f\pi^2) + 0.0[illegible]\right]$$

$$+ 0.00091800\,(f\pi^2)^2$$

$$\pi^2\frac{\sigma}{E} = \lambda^2\left(\frac{a}{R}\right)^2\left\{0.19282196\,\pi^4 f^2 - 0.41265255\,\pi^2 f + 0.55068290\right\}\frac{1}{6}$$

$$+ \frac{\left(\frac{t}{R}\right)^2}{\left(\frac{a}{R}\right)^2}\left\{0.93370370\right\}\pi^4$$

$$K = \beta^2\left\{0.04627727\,\pi^2 f^2 - 0.09903661\,f + \frac{0.13216390}{\pi^2}\right\} + \frac{0.93370370\,\pi^2}{\beta^2}$$

$$K = \frac{\pi^2}{g^2}\left\{0.18510908\left(\frac{\delta}{t}\right)^2 + 0.93370370\right\} + \frac{0.13216390}{\pi^2}g^2$$
$$- 0.19807322\left(\frac{\delta}{t}\right)$$

$$= 2\left\{0.02446474\left(\frac{\delta}{t}\right)^2 + 0.12340192\right\}^{\frac{1}{2}} - 0.19807322\left(\frac{\delta}{t}\right)$$

$K_0 = 0.7026$

$$0.04892948\left(\frac{\delta}{t}\right) = 0.19807322\left\{0.02446474\left(\frac{\delta}{t}\right)^2 + 0.12340192\right\}$$

$0.002394094\left(\frac{\delta}{t}\right)^2 = 0.00484143$

0.000959825

$\left(\frac{\delta}{t}\right)^2 = 3.375538$

$\left(\frac{\delta}{t}\right) = 1.837264$

$K_{min.} = 0.5438$

①	②	③	④	⑤	⑥	⑦
(δ/t)	0.024464 ①^2	② $+0.12340$	$2\times$ ③$^{\frac{1}{2}}$	K		
1	0.024464	0.14786	0.7690	0.5709		
2	0.097856	0.22126	0.9408	0.5447		
3	0.220176	0.34358	1.1724	0.5382		
4	0.391424	0.51482	1.4350	0.6422		
5	0.611600	0.73500	1.7146	0.7243		

$$\lambda = 0.8000$$

$$F_3 = \left\{0.11426544 + 0.02323766 + 0.000725745 + 0.00123288\right\}(f\pi^2)^2$$

$$-\left\{0.61035156 + 0.09295062\right\}(f\pi^2) + \left\{1.22070313 + 0.09295062\right\}$$

$$= 0.13946173\,(f\pi^2)^2 - 0.70330218\,(f\pi^2) + 1.31365375$$

$$H_1(\lambda) = 0.25 + 0.18590125 + 0.01972604 = 0.45562729$$

$$H_2(\lambda) = 0.37180250\,, \quad H_3(\lambda) = 0.0625 + 0.04644768 = 0.10894768$$

$$-\left[0.45562729\,(f\pi^2) - 0.37180250\right]\left[0.1225400\,(f\pi^2) - 0.07957143\right]$$

$$-\ 0.01631338 4 \times 0.10894768\,(f\pi^2)^2$$

$$= -\ 0.05760987\,(f\pi^2)^2 + 0.08181559\,(f\pi^2) - 0.02961491 = F_4$$

$$F_2 = 0.13302164\left[0.45562729\,(f\pi^2) - 0.37180250\right]^2$$

$$+\ 0.01654153 \times (0.10894768)^2\,(f\pi^2)^{-1}$$

$$= 0.13302164\left\{0.20759623\,(f\pi^2)^2 - 0.33880673\,(f\pi^2) + 0.13823710\right\}$$

$$+\ 0.00157891\,(f\pi^2)^2$$

$$\pi^2\frac{\sigma}{E} = \left(\frac{a}{R}\right)^2\left\{0.28427664\,\pi^4 f^2 - 1.27978602\,\pi^2 f + 1.66710703\right\}\frac{1}{6}$$

$$+\ \frac{\left(\frac{t}{R}\right)^2}{\left(\frac{a}{R}\right)^2}\left\{0.95638889\right\}\pi^4$$

$$K = \gamma^2\left\{0.04737944\,\pi^2 f^2 - 0.21329767\,f + \frac{0.277851172}{\pi^2}\right\} + \frac{\pi^2}{\gamma^2}\,0.95638889 \qquad \underline{\underline{499}}$$

$$= \frac{\pi^2}{\gamma^2}\left\{0.18951776\left(\frac{\delta}{t}\right)^2 + 0.95638889\right\} + \frac{0.277851172\,\gamma^2}{\pi^2} - 0.42659534\left(\frac{\delta}{t}\right)$$

$$= 2\left\{0.05265773\left(\frac{\delta}{t}\right)^2 + 0.26573377\right\}^{\frac{1}{2}} - 0.42659534\left(\frac{\delta}{t}\right)$$

$$K_0 = 1.0310$$

$$0.10531546\left(\frac{\delta}{t}\right) = 0.42659534\left(0.05265773\left(\frac{\delta}{t}\right)^2 + 0.26573377\right)^{\frac{1}{2}}$$

$$\begin{array}{l} 0.011091346 \\ \underline{0.009582842} \\ 0.001508504 \end{array}\left(\frac{\delta}{t}\right)^2 = 32.05771 \qquad \left(\frac{\delta}{t}\right) = 5.66195$$

$$K_{min} = 0.3802$$

①	②	③	④	⑤	⑥		
(δ/t)							
1							
2							
3							
4							
5							
6							
7							
8							
9							
10							

$$\lambda = 0.600$$ <u>500</u>

$$F_3 = \left\{ 0.28940008 + 0.03379109 + 0.00082195 + 0.00262446 \right\} (f\pi^2)^2$$

$$- \left\{ 1.92901235 + 0.13516436 \right\} (f\pi^2) + \left\{ 3.85802469 + 0.13516436 \right\}$$

$$= 0.32663758 (f\pi^2)^2 - 2.06417671 (f\pi^2) + 3.99318905$$

$$H_1(\lambda) \doteq 0.25 + 0.27032872 + 0.04199140 = 0.56232012, \quad H_2(\lambda) = 0.54065744$$

$$H_3(\lambda) = 0.0625 + 0.05260500 = 0.11510500$$

$$F_1 = -\left[0.56232012 (f\pi^2) - 0.54065744 \right]\left[0.12254000 (f\pi^2) - 0.07957143 \right]$$

$$- 0.11510500 \times 0.01631338 4 (f\pi^2)^2$$

$$= -0.07078446 (f\pi^2)^2 + 0.11099678 (f\pi^2) - 0.04302089$$

$$F_2 = 0.11619690 \left[0.56232012 (f\pi^2) - 0.54065744 \right]^2$$

$$+ (0.11510500)^2 (f\pi^2)^2 \times 0.05818125$$

$$= 0.11619690 \left[0.31620392 (f\pi^2)^2 - 0.60804511 (f\pi^2) + 0.29231042 \right] + 0.00077092 (f\pi^2)^2$$

$$K = \gamma^2 \left\{ 0.07040783 \pi^2 f^2 - 0.36428992 f + \frac{0.47809605}{\pi^2} \right\} + \frac{\pi^2}{\gamma^2} 1.26814815$$

$$= \frac{\pi^2}{\gamma^2} \left\{ 0.28163132 \left(\frac{\delta}{t}\right)^2 + 1.26814815 \right\} + \frac{0.47809605 \gamma^2}{\pi^2} - 0.72857984 \left(\frac{\delta}{t}\right)$$

$$= 2 \left\{ 0.13464682 \left(\frac{\delta}{t}\right)^2 + 0.60629662 \right\}^{\frac{1}{2}} - 0.72857984 \left(\frac{\delta}{t}\right)$$

$$K_0 = 1.55730$$

$$0.26929364\left(\frac{\delta}{t}\right) = 0.72857984\left\{0.13464682\left(\frac{\delta}{t}\right)^2 + 0.60629662\right\}^{\frac{1}{2}}$$ 501

$$\begin{array}{r} 0.072519065 \\ \underline{0.071474380} \\ 0.001044685 \end{array} \Bigg) \left(\frac{\delta}{t}\right)^2 = 308.0733 \qquad \left(\frac{\delta}{t}\right) = 17.55202$$

$$\underline{\underline{K_{min} = 0.1869}}$$

①	②	③	④	⑤	⑥		
(δ/t)							
2							
4							
6							
8							
10							
12							
14							
16							
18							
20							
22							

50.2

Consider the limiting case, when $\lambda << 1$.

then
$$6\pi^2 \frac{\sigma}{E} = \frac{1}{\lambda^2}\left[\frac{17}{128} f^2 \pi^4 - \frac{3}{4} f \pi^2 + 1\right]\left(\frac{a}{R}\right)^2 + \frac{\left(\frac{t}{R}\right)^2}{\left(\frac{a}{R}\right)^2}\, 2 \frac{1}{\lambda^2} \pi^4$$

or
$$\lambda^2 K = \gamma^2 \left[\frac{17}{768} \pi^2 f^2 - \frac{1}{8} f + \frac{1}{6\pi^2}\right] + \frac{\pi^2}{\gamma^2} \frac{1}{3}$$

$$= \frac{\pi^2}{\gamma^2}\left[\frac{17}{192}\left(\frac{\delta}{t}\right)^2 + \frac{1}{3}\right] + \frac{1}{6}\frac{\gamma^2}{\pi^2} - \frac{1}{4}\left(\frac{\delta}{t}\right)$$

$$= 2\left[\frac{17}{1152}\left(\frac{\delta}{t}\right)^2 + \frac{1}{18}\right]^{\frac{1}{2}} - \frac{1}{4}\left(\frac{\delta}{t}\right) \qquad \underline{\text{O.K.}}$$

$$\frac{17}{1152}\left(\frac{\delta}{t}\right)^2 + \frac{1}{18} = \frac{1}{64}\left(\frac{\delta}{t}\right)^2$$

$$\frac{1}{1152}\left(\frac{\delta}{t}\right)^2 = \frac{1}{18} \qquad \left(\frac{\delta}{t}\right) = \sqrt{\frac{1152}{18}} = 8$$

$$\gamma^2_{min} = \pi^2\left[34 + 2\right]^{\frac{1}{2}} = \pi^2 36 \qquad \gamma = 6^{\frac{1}{2}}\pi$$

$$\lambda = 0.4$$

$$F_3 = \left\{1.33020020 + 0.04644768 + 0.00090289 + 0.00580941\right\}(f\pi^2)^2$$

$$-\left\{9.76562500 + 0.18579073\right\}(f\pi^2) + \left\{19.5312500 + 0.18579073\right\}$$

$$= 1.38336018\,(f\pi^2)^2 - 9.95141573\,(f\pi^2) + 19.71704073$$

$$H_1(\lambda) = 0.25 + 0.37158145 + 0.09295062 = 0.71453207, \quad H_2(\lambda) = 0.74316290$$

$$H_3(\lambda) = 0.0625 + 0.05778476 = 0.12028476$$

$$F_1 = -\left[0.71453207(f\pi^2) - 0.74316290\right]\left[0.1225400\,(f\pi^2) - 0.07957143\right]$$

$$- 0.12028426 \times 0.016313384\,(f\pi^2)^2$$

$$= -0.08952101\,(f\pi^2)^2 + 0.14792352\,(f\pi^2) - 0.05913453$$

$$F_2 = 0.09004850\left[0.71453207(f\pi^2) - 0.74316290\right]^2 + 0.04515168 \times (0.12028426)^2 (f\pi^2)^2$$

$$= 0.09004850\left[0.51055608\,(f\pi^2)^2 - 1.06202725\,(f\pi^2) + 0.55229110\right]$$

$$+ 0.00065327\,(f\pi^2)^2$$

$$K = \gamma^2\left\{0.14298\ldots\,\pi^2 f^2 - 0.29193010 f + \frac{1.05107409}{\pi^2}\right\} + \frac{\pi^2}{\gamma^2}\,2.35888669$$

$$= \frac{\pi^2}{\gamma^2}\left\{0.57193268\left(\frac{\delta}{t}\right)^2 + 2.35888669\right\} + \frac{1.05107409\,\gamma^2}{\pi^2} - 1.58386020\left(\frac{\delta}{t}\right)$$

$$= 2\left\{0.60114362\left(\frac{\delta}{t}\right)^2 + 2.47936699\right\}^{\frac{1}{2}} - 1.58386020\left(\frac{\delta}{t}\right)$$

$$K_0 = 3.14920,$$

$$\gamma^2 = \pi^2\Big\{$$

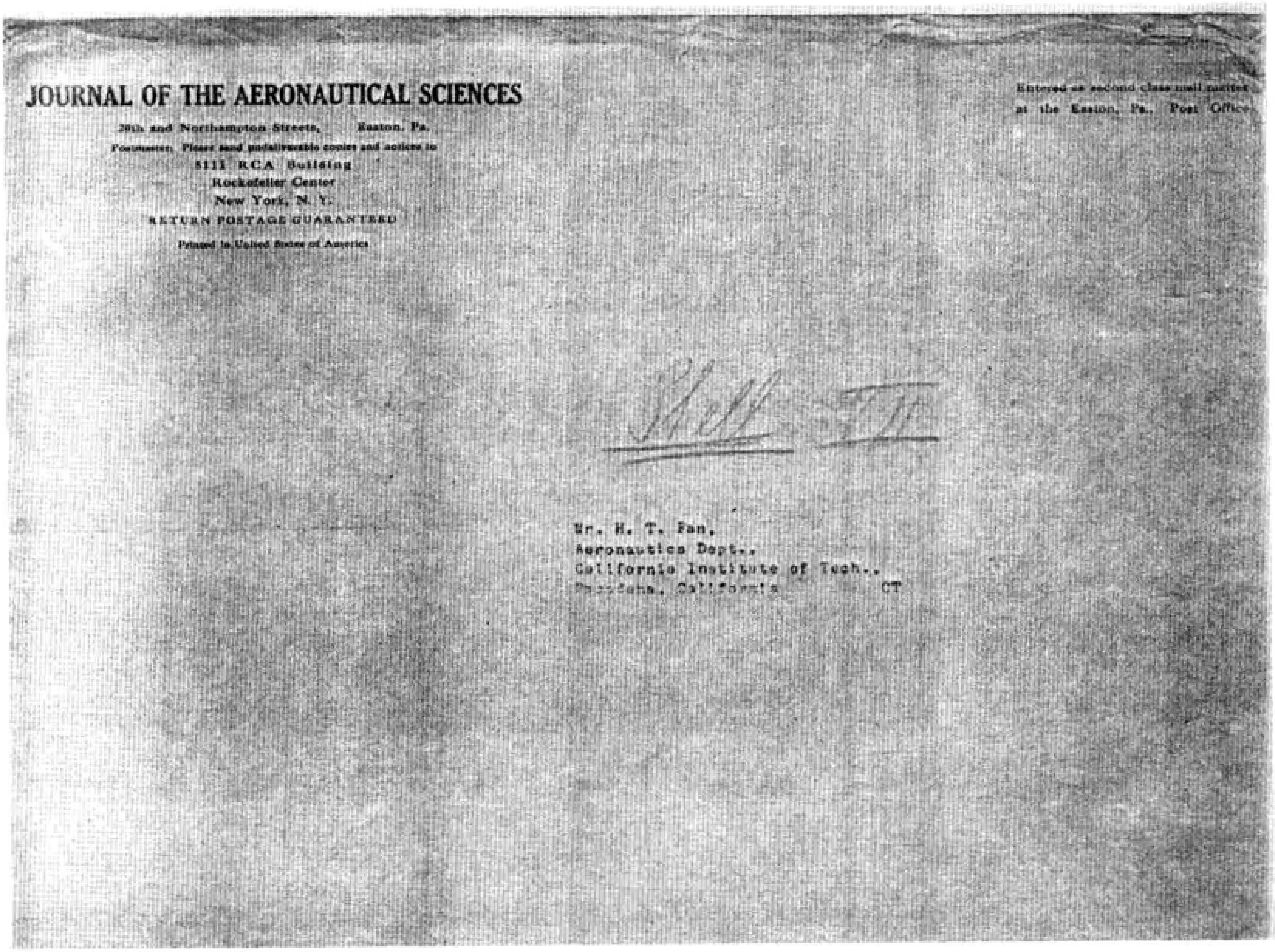
$$\cancel{1.20228724\left(\frac{\delta}{t}\right) = 1.58\ldots\left\{0.60114362\left(\frac{\delta}{t}\right)^2 + 2.4793\ldots\right\}}^{\frac{1}{2}}$$

$1.\cancel{44549461}$

$$0.60114362\left(\frac{\delta}{t}\right)^2 + 2.47936699 = 0.62715328\left(\frac{\delta}{t}\right)^2$$

$$\begin{array}{r} 0.62715328 \\ 0.60114362 \\ \hline 0.02600966 \end{array}$$

$$\left(\frac{\delta}{t}\right)_0 = 9.7588$$



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$$\frac{w}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left[1-\left(\frac{x}{a}\right)^2 - \frac{f}{4}\left(1+\cos\frac{\pi x}{a}\right)\left(1+\cos\frac{\pi y}{b}\right)\right]$$

$$\frac{w_0}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left[1-\left(\frac{x}{a}\right)^2\right]$$

$$\sigma_x = E\left\{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right\}$$

$$\sigma_y = E\left\{\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\}$$

$$\tau_{xy} = E\left\{\frac{1}{2}\frac{\partial v}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right\}$$

By using the equilibrium equation: $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$, we have

$$\frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{1}{2}\left\{\frac{\partial w}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 w}{\partial y^2}\right) + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\right\}$$

We have

$$\frac{\partial w}{\partial x} = \left(\frac{a}{R}\right)\left\{-\left(\frac{x}{a}\right) + \frac{f}{8}\pi \sin\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)\right\}$$

$$\frac{\partial w}{\partial y} = \left(\frac{a}{R}\right)\frac{f}{8}\pi\left(\frac{a}{b}\right)\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{R}\left[-1 + \frac{f}{8}\pi^2\cos\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)\right\}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{R}\left[\frac{f}{8}\pi^2\left(\frac{a}{b}\right)^2\left(1+\cos\frac{\pi x}{a}\right)\cos\frac{\pi y}{b}\right]$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{R}\left[-\frac{f}{8}\pi^2\left(\frac{a}{b}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right]$$

$$R\left[\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial y^2}\right]=-\left(\frac{a}{R}\right)\frac{f}{g}\pi\lambda\left[\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}\left\{\frac{f}{g}\pi^2\cos\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)+\frac{f}{g}\pi^2\left(\frac{a}{b}\right)^2 2\left(1+\cos\frac{\pi x}{a}\right)\cos\frac{\pi y}{b}-1\right\}\right.$$

$$\left.-\pi\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\left\{\frac{f}{g}\pi\sin\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)-\frac{x}{a}\right\}\right]$$

$$=-\left(\frac{a}{R}\right)\frac{f}{g}\pi\lambda\left[\frac{f\pi^2}{\cancel{32}\,16}\left(\cancel{1}+\cancel{2}\cos\frac{\pi x}{a}+\cancel{}\cos\frac{2\pi x}{a}\right)\left(2\sin\frac{\pi y}{b}+\sin\frac{2\pi y}{b}\right)+\frac{f\pi^2}{16}\left(\frac{a}{b}\right)^2\left(3+4\cos\frac{\pi x}{a}+\cos\frac{2\pi x}{a}\right)\sin\frac{2\pi y}{b}\right.$$

$$-\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}$$

$$\left.-\cancel{\frac{f\pi^2}{32}\left(1-\cos\frac{2\pi x}{a}\right)\left(2\sin\frac{\pi y}{b}+\sin\frac{2\pi y}{b}\right)}+\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right]$$

$$=-\left(\frac{a}{R}\right)\frac{f}{g}\pi\lambda\left[\frac{f\pi^2}{8}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}+\frac{f\pi^2}{8}\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}+\frac{f\pi^2}{16}\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}+\frac{f\pi^2}{16}\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\right.$$

$$+\frac{f\pi^2}{16}\left(\frac{a}{b}\right)^2 3\sin\frac{2\pi y}{b}+\frac{f\pi^2}{4}\left(\frac{a}{b}\right)^2\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}+\frac{f\pi^2}{16}\left(\frac{a}{b}\right)^2\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}$$

$$\left.-\sin\frac{\pi y}{b}-\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}+\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right]$$

$$R\left[\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial y^2}\right] = -\left(\frac{q}{R}\right)\frac{f}{8}\pi\lambda\left[-\sin\frac{\pi y}{b}+\frac{3}{16}(f\pi^2)\lambda^2\sin\frac{2\pi y}{b}+\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}\right.$$

$$+\frac{f\pi^2}{8}\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}+\frac{f\pi^2}{16}(1+4\lambda^2)\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}+\frac{f\pi^2}{16}(1+\lambda^2)\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}$$

$$\left.+\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right]$$

The particular integral is

$$R\,v = +\left(\frac{q}{R}\right)\frac{f}{8}\pi\lambda\left[-\frac{\sin\frac{\pi y}{b}}{2\left(\frac{\pi}{b}\right)^2}+\frac{3}{16}(f\pi^2)\lambda^2\frac{\sin\frac{2\pi y}{b}}{8\left(\frac{\pi}{b}\right)^2}+\left(\frac{f\pi^2}{8}-1\right)\frac{\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}}{\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2}\right.$$

$$+\frac{f\pi^2}{8}\frac{\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}}{4\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2}+\frac{f\pi^2}{16}(1+4\lambda^2)\frac{\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}}{\left(\frac{\pi}{a}\right)^2+8\left(\frac{\pi}{b}\right)^2}+\frac{f\pi^2}{16}(1+\lambda^2)\frac{\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}}{4\left(\frac{\pi}{a}\right)^2+8\left(\frac{\pi}{b}\right)^2}$$

$$\left.+\frac{2\left(\frac{\pi}{a}\right)^2}{\left[\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2\right]^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}+\frac{1}{\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right]$$

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$$\frac{v}{R} = \left(\frac{a}{R}\right)^3 \frac{f}{8} \frac{1}{\pi} \lambda \Big[-\frac{1}{2\lambda^2} \sin\frac{\pi y}{b} + \frac{3}{128}(f\pi^2) \sin\frac{2\pi y}{b} + \left(\frac{f\pi^2}{8} - 1\right) \frac{\cos\frac{\pi x}{a} \sin\frac{\pi y}{b}}{1+2\lambda^2}$$

$$+ \frac{f\pi^2}{16} \frac{\cos\frac{2\pi x}{a} \sin\frac{\pi y}{b}}{2+\lambda^2} + \frac{f\pi^2}{16}\left(\frac{1+4\lambda^2}{1+8\lambda^2}\right) \cos\frac{\pi x}{a} \sin\frac{2\pi y}{b} + \frac{f\pi^2}{64}\left(\frac{1+\lambda^2}{1+2\lambda^2}\right) \cos\frac{2\pi x}{a} \sin\frac{2\pi y}{b}$$

$$+ \frac{2\lambda^2}{(1+2\lambda^2)^2} \cos\frac{\pi x}{a} \sin\frac{\pi y}{b} + \frac{1}{1+2\lambda^2} \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + a_0\left(\frac{\pi y}{b}\right) + a_1 \cosh\frac{\sqrt{2}\lambda\pi x}{a} \sin\frac{\pi y}{b} \Big]$$

$$\frac{\partial v}{\partial x} = \left(\frac{a}{R}\right)^2 \frac{f}{8} \lambda \Big[-\left\{\frac{f\pi^2}{8}(1+\lambda^2) - 1\right\} \frac{\sin\frac{\pi x}{a} \sin\frac{\pi y}{b}}{(1+2\lambda^2)^2} - \frac{f\pi^2}{8} \frac{\sin\frac{2\pi x}{a} \sin\frac{\pi y}{b}}{(2+\lambda^2)} - \frac{f\pi^2}{16}\left(\frac{1+4\lambda^2}{1+8\lambda^2}\right) \sin\frac{\pi x}{a} \sin\frac{2\pi y}{b}$$

$$- \frac{f\pi^2}{32}\left(\frac{1+\lambda^2}{1+2\lambda^2}\right) \sin\frac{2\pi x}{a} \sin\frac{2\pi y}{b} + \frac{1}{1+2\lambda^2} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + \frac{1}{1+2\lambda^2} \left(\frac{\pi x}{a}\right) \cos\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$+ a_1 \sqrt{2}\lambda \sinh\frac{\sqrt{2}\lambda\pi x}{a} \sin\frac{\pi y}{b} \Big]$$

$$\boxed{a_1 = \frac{\pi}{(1+2\lambda^2)\sqrt{2}\lambda \sinh\sqrt{2}\lambda\pi} = \frac{1}{2\lambda^2(1+2\lambda^2)} \frac{1}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)}}$$

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$$\frac{\partial v}{\partial y}=\left(\frac{a}{R}\right)^2\frac{f}{8}\left[-\frac{1}{2}\cos\frac{\pi y}{b}+\frac{3\lambda^2}{64}(f\pi^2)\cos\frac{2\pi y}{b}+\frac{\lambda^2}{1+2\lambda^2}\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}\right.$$

$$+\frac{\lambda^2}{2+\lambda^2}\frac{f\pi^2}{16}\cos\frac{2\pi x}{a}\cos\frac{\pi y}{b}+\frac{\lambda^2(1+4\lambda^2)}{1+8\lambda^2}\frac{f\pi^2}{8}\cos\frac{\pi x}{a}\cos\frac{2\pi y}{b}+\frac{\lambda^2(1+\lambda^2)}{1+2\lambda^2}\frac{f\pi^2}{32}\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}$$

$$\left.+\frac{2\lambda^4}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}+\frac{\lambda^2}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}+a_0\lambda^2+a_1\lambda^2\cosh\frac{\sqrt{2}\lambda\pi x}{a}\cos\frac{\pi y}{b}\right]$$

$$\left(\frac{\partial v}{\partial y}\right)_{y=b}=\left(\frac{a}{R}\right)^2\frac{f}{8}\left[\frac{1}{2}+\frac{3\lambda^2}{64}(f\pi^2)-\frac{\lambda^2}{1+2\lambda^2}\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi x}{a}-\frac{\lambda^2}{2+\lambda^2}\frac{f\pi^2}{16}\cos\frac{2\pi x}{a}\right.$$

$$+\frac{\lambda^2(1+4\lambda^2)}{1+8\lambda^2}\frac{f\pi^2}{8}\cos\frac{\pi x}{a}+\frac{\lambda^2(1+\lambda^2)}{1+2\lambda^2}\frac{f\pi^2}{32}\cos\frac{2\pi x}{a}-\frac{2\lambda^4}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a}-\frac{\lambda^2}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)$$

$$\left.+a_0\lambda^2-a_1\lambda^2\cosh\frac{\sqrt{2}\lambda\pi x}{a}\right]$$

$$-\frac{\sigma}{E}=\left(\frac{a}{R}\right)^2\frac{f}{8}\left[\frac{1}{2}+\frac{3\lambda^2}{64}(f\pi^2)-\frac{\lambda^2}{1+2\lambda^2}+a_0\lambda^2-\frac{a_1\lambda^2}{\sqrt{2}\lambda\pi}\sinh\sqrt{2}\lambda\pi\right]$$

$$=\left(\frac{a}{R}\right)^2\frac{f}{8}\left[\frac{1}{2}+\frac{3}{64}\lambda^2(f\pi^2)-\frac{\lambda^2}{1+2\lambda^2}+a_0\lambda^2-\frac{1}{2(1+2\lambda^2)}\right]$$

$$=\left(\frac{a}{R}\right)^2\frac{f}{8}\left[\frac{3}{64}\lambda^2(f\pi^2)+a_0\lambda^2\right]$$

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$$\left(\frac{a}{R}\right)^2 \frac{f}{8} a_0 \lambda^2 = -\frac{\sigma}{E} - \left(\frac{a}{R}\right)^2 \frac{f}{8} \frac{3}{64} \lambda^2 (f\pi^2)$$

$$\boxed{\frac{\Delta p}{E a b t} = -\left(\frac{\sigma}{E}\right)^2 - \left(\frac{a}{R}\right)^2 \left(\frac{f}{8}\right)^2 \left(\frac{3}{8} \lambda^2 \pi^2\right) \frac{\sigma}{E}.}$$

$$\frac{\partial v}{\partial y} = \left(\frac{a}{R}\right)^2 \frac{f}{8} \Big[-\frac{1}{2} \cos\frac{\pi y}{b} + \cancel{\frac{3}{64} \lambda^2 (f\pi^2) \cos\frac{2\pi y}{b}} + \frac{\lambda^2}{1+2\lambda^2}\left(\frac{f\pi^2}{8} - 1\right) \cos\frac{\pi x}{a} \cos\frac{\pi y}{b}$$

$$+ \frac{\lambda^2}{9+\lambda^2} \frac{f\pi^2}{16} \cos\frac{2\pi x}{a} \cos\frac{\pi y}{b} + \frac{\lambda^2(1+4\lambda^2)}{1+8\lambda^2} \frac{f\pi^2}{8} \cos\frac{\pi x}{a} \cos\frac{2\pi y}{b} + \frac{\lambda^2(1+\lambda^2)}{1+2\lambda^2} \frac{f\pi^2}{32} \cos\frac{2\pi x}{a} \cos\frac{2\pi y}{b}$$

$$+ \frac{2\lambda^4}{(1+2\lambda^2)^2} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} + \frac{\lambda^2}{1+2\lambda^2} \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a} \cos\frac{\pi y}{b} - \frac{3}{64} \lambda^2 (f\pi^2) + a_1 \lambda^2 \cosh\frac{\sqrt{2}\lambda\pi x}{a} \cos\frac{\pi y}{b} \Big] - \frac{\sigma}{E}$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{a}{R}\right)^2 \frac{f}{8} \Big[\frac{f\pi^2}{64} \lambda^2 \Big\{ 3 + 4\cos\frac{\pi x}{a} + \cos\frac{2\pi x}{a} - 3\cancel{\cos\frac{2\pi y}{b}} - 4\cos\frac{\pi x}{a} \cos\frac{2\pi y}{b} - \cos\frac{2\pi x}{a} \cos\frac{2\pi y}{b} \Big\} \Big]$$

$$\frac{\sigma_y}{E} = \left(\frac{a}{R}\right)^2 \frac{f}{8} \Big[\Big\{ \frac{f\pi^2}{16} \lambda^2 \cos\frac{\pi x}{a} + \frac{f\pi^2}{64} \lambda^2 \cos\frac{2\pi x}{a} \Big\}$$

$$+ \Big\{ -\frac{1}{2} + \left(\frac{f\pi^2}{8} \frac{\lambda^2}{1+2\lambda^2} - \frac{\lambda^2}{(1+2\lambda^2)^2} \right) \cos\frac{\pi x}{a} + \frac{\lambda^2}{9+\lambda^2} \frac{f\pi^2}{16} \cos\frac{2\pi x}{a} + \frac{\lambda^2}{1+2\lambda^2} \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a} + a_1 \lambda^2 \cosh\frac{\sqrt{2}\lambda\pi x}{a} \Big\} \cos\frac{\pi y}{b}$$

$$+ \Big\{ \frac{\lambda^2}{1+8\lambda^2} \frac{f\pi^2}{16} \cos\frac{\pi x}{a} + \frac{\lambda^2}{1+9\lambda^2} \frac{f\pi^2}{64} \cos\frac{2\pi x}{a} \Big\} \cos\frac{2\pi y}{b} \Big] - \frac{\sigma}{E}$$

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$$\frac{1}{2E^2ab}\int_0^a\int_0^b \sigma_y^2\,dx\,dy = \left(\frac{a}{h}\right)^4\left(\frac{f}{8}\right)^2\Big[\frac{1}{4}\left(\frac{f\pi^2\lambda^2}{16}\right)^2+\left(\frac{f\pi^2}{64}\lambda^2\right)^2\frac{1}{4} + \frac{1}{8}\left(\frac{\lambda^2}{1+8\lambda^2}\frac{f\pi^2}{16}\right)^2+\frac{1}{8}\left(\frac{\lambda^2}{1+2\lambda^2}\frac{f\pi^2}{64}\right)^2$$

$$+\frac{1}{16}+\frac{1}{8}\left(\frac{f\pi^2}{8}\frac{\lambda^2}{1+2\lambda^2}-\frac{\lambda^2}{(1+2\lambda^2)^2}\right)^2+\frac{1}{8}\left(\frac{\lambda^2}{2+\lambda^2}\frac{f\pi^2}{16}\right)^2$$

$$-\frac{1}{4}\int_0^1\left\{\frac{\lambda^2}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}+a_1\lambda^2\cosh\frac{\sqrt{2}\lambda\pi x}{a}\right\}d\left(\frac{x}{a}\right)$$

$$+\frac{1}{2}\left\{\frac{f\pi^2}{8}\frac{\lambda^2}{1+2\lambda^2}-\frac{\lambda^2}{(1+2\lambda^2)^2}\right\}\int_0^1\cos\frac{\pi x}{a}\left\{\frac{\lambda^2}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}+a_1\lambda^2\cosh\frac{\sqrt{2}\lambda\pi x}{a}\right\}d\left(\frac{x}{a}\right)$$

$$+\frac{1}{2}\frac{\lambda^2}{2+\lambda^2}\frac{f\pi^2}{16}\int_0^1\cos\frac{2\pi x}{a}\left\{\frac{\lambda^2}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}+a_1\lambda^2\cosh\frac{\sqrt{2}\lambda\pi x}{a}\right\}d\left(\frac{x}{a}\right)$$

$$+\frac{1}{4}\int_0^1\left\{\frac{\lambda^2}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}+a_1\lambda^2\cosh\frac{\sqrt{2}\lambda\pi x}{a}\right\}^2 d\left(\frac{x}{a}\right)\Big]$$

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$$\frac{1}{2E^2ab}\int_0^a\int_0^b \sigma_y^2\,dx\,dy = \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\Bigg[\frac{17}{16384}(f\pi^2)^2\lambda^4 + \frac{1}{2048}\left(\frac{\lambda^2}{1+8\lambda^2}\right)^2(f\pi^2)^2 + \frac{1}{32768}\left(\frac{\lambda^2}{1+2\lambda^2}\right)^2(f\pi^2)^2 + \frac{1}{16}$$

$$+\frac{1}{512}\left(\frac{\lambda^2}{1+2\lambda^2}\right)^2(f\pi^2)^2 + \frac{1}{2048}\left(\frac{\lambda^2}{2+\lambda^2}\right)^2(f\pi^2)^2 - \frac{1}{32}\frac{\lambda^4}{(1+2\lambda^2)^3}(f\pi^2) + \frac{1}{8}\frac{\lambda^4}{(1+2\lambda^2)^4}$$

$$-\frac{1}{4}\frac{\lambda^2}{1+2\lambda^2} - \frac{1}{4}\frac{1}{2(1+2\lambda^2)} + \left\{\frac{f\pi^2}{8}\frac{\lambda^2}{1+2\lambda^2} - \frac{\lambda^2}{(1+2\lambda^2)^2}\right\}\left\{-\frac{1}{8}\frac{\lambda^2}{1+2\lambda^2} - \frac{1}{2}\frac{\lambda^2}{(1+2\lambda^2)^2}\right\}$$

$$+\frac{\lambda^2}{2+\lambda^2}\frac{f\pi^2}{32}\left\{-\frac{1}{3}\frac{\lambda^2}{1+2\lambda^2} + \frac{\lambda^2}{(1+2\lambda^2)(2+\lambda^2)}\right\} + \frac{1}{4}\left(\frac{\pi^2}{6}-\frac{1}{4}\right)\left(\frac{\lambda^2}{1+2\lambda^2}\right)^2$$

$$+\frac{\lambda^2}{8(1+2\lambda^2)^2}\frac{1}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)}\left\{\frac{\cosh\sqrt{2}\lambda\pi}{1+2\lambda^2} - \frac{4\lambda^2}{(1+2\lambda^2)^2}\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)\right\}$$

$$+\frac{1}{4}\frac{1}{4(1+2\lambda^2)^2}\frac{1}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)^2}\left\{\frac{1}{2}+\frac{1}{4\sqrt{2}\lambda\pi}\sinh 2\sqrt{2}\lambda\pi\right\}\Bigg] - \frac{1}{2}\left(\frac{\sigma}{E}\right)^2$$

$$\frac{1}{2E^2ab}\int_0^a\int_0^b \sigma_x^2\,dx\,dy = \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\left[\frac{105}{32768}(f\pi^2)^2 - \frac{5}{48}(f\pi^2) + \left(\frac{\pi^2}{8}-\frac{3}{16}\right)\right]$$

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$$\frac{\partial v}{\partial x} = \left(\frac{a}{R}\right)^2 \frac{f}{8}\lambda \Bigg[\left\{ 2\frac{(1+\lambda^2)}{(1+2\lambda^2)^2} - \frac{1}{1+2\lambda^2}\frac{f\pi^2}{8} \right\} \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} - \frac{1}{8(2+\lambda^2)}(f\pi^2)\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b}$$

$$- \left(\frac{1+4\lambda^2}{1+8\lambda^2}\right)\frac{f\pi^2}{16}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b} - \left(\frac{1+\lambda^2}{1+2\lambda^2}\right)\frac{f\pi^2}{32}\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b} + \frac{1}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$+ a_1\sqrt{2}\lambda\sinh\frac{\sqrt{2}\lambda\pi x}{a}\sin\frac{\pi y}{b}\Bigg]$$

$$\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \left(\frac{a}{R}\right)^2\frac{f\lambda}{8}\left\{ -\left(\frac{\pi x}{a}\right)\sin\frac{\pi y}{b} - \left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{f\pi^2}{32}\left(2\sin\frac{\pi x}{a} + \sin\frac{2\pi x}{a}\right)\left(2\sin\frac{\pi y}{b} + \sin\frac{2\pi y}{b}\right)\right\}$$

$$= \left(\frac{a}{R}\right)^2\frac{f\lambda}{8}\left\{ -\left(\frac{\pi x}{a}\right)\sin\frac{\pi y}{b} - \left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{f\pi^2}{8}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{f\pi^2}{16}\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b}\right.$$

$$\left. + \frac{f\pi^2}{16}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b} + \frac{f\pi^2}{32}\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\right\}$$

$$\frac{2\tau_{xy}}{E} = \left(\frac{a}{R}\right)^2\frac{f\lambda}{8}\left\{ -\left(\frac{\pi x}{a}\right)\sin\frac{\pi y}{b} - \frac{2\lambda^2}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + 2\left\{\frac{1+\lambda^2}{(1+2\lambda^2)^2} + \frac{\lambda^2}{1+2\lambda^2}\frac{f\pi^2}{8}\right\}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right.$$

$$+ \frac{\lambda^2}{16(2+\lambda^2)}(f\pi^2)\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b} + \frac{\lambda^2}{4(1+8\lambda^2)}(f\pi^2)\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b} + \frac{\lambda^2}{(1+2\lambda^2)}\frac{f\pi^2}{32}\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b}$$

$$\left. + \frac{a_1}{2}\sqrt{2}\lambda\sinh\frac{\sqrt{2}\lambda\pi x}{a}\sin\frac{\pi y}{b}\right\}$$

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$$\frac{\tau_{xy}}{E} = \left(\frac{a}{R}\right)^2 \frac{f}{8}\Bigg[\Bigg\{-\frac{\lambda}{2}\left(\frac{\pi x}{a}\right) - \frac{\lambda^3}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a} + \left\{\frac{\lambda(1+\lambda^2)}{(1+2\lambda^2)^2} + \frac{\lambda^3}{1+2\lambda^2}\frac{f\pi^2}{8}\right\}\sin\frac{\pi x}{a} + \frac{\lambda^3}{32(2+\lambda^2)}(f\pi^2)\sin\frac{2\pi x}{a} + \frac{a_1}{2}\sqrt{2}\lambda^2\sinh\frac{\sqrt{2}\lambda\pi x}{a}\Bigg\}\sin\frac{\pi y}{b}$$

$$+\left\{\frac{\lambda^3}{8(1+8\lambda^2)}(f\pi^2)\sin\frac{\pi x}{a} + \frac{\lambda^3}{(1+2\lambda^2)}\frac{f\pi^2}{64}\sin\frac{2\pi x}{a}\right\}\sin\frac{2\pi y}{b}\Bigg]$$

$$\frac{1}{E^2ab}\int_0^a\int_0^b \tau_{xy}^2\,dx\,dy = \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\Bigg[\frac{1}{4}\left\{\frac{\lambda(1+\lambda^2)}{(1+2\lambda^2)^2} + \frac{\lambda^3}{1+2\lambda^2}\frac{f\pi^2}{8}\right\}^2 + \frac{1}{4}\left\{\frac{\lambda^3}{32(2+\lambda^2)}f\pi^2\right\}^2$$

$$+\frac{1}{4}\left\{\frac{\lambda^3}{8(1+8\lambda^2)}f\pi^2\right\}^2 + \frac{1}{4}\left\{\frac{\lambda^3}{1+2\lambda^2}\frac{f\pi^2}{64}\right\}^2$$

$$+\left\{\frac{\lambda(1+\lambda^2)}{(1+2\lambda^2)^2} + \frac{\lambda^3}{1+2\lambda^2}\frac{f\pi^2}{8}\right\}\int_0^1 \sin\frac{\pi x}{a}\left\{-\frac{\lambda}{2}\left(\frac{\pi x}{a}\right) - \frac{\lambda^3}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a} + \frac{a_1}{2}\sqrt{2}\lambda^2\sinh\frac{\sqrt{2}\lambda\pi x}{a}\right\}d\left(\frac{x}{a}\right)$$

$$+\frac{\lambda^3}{32(2+\lambda^2)}(f\pi^2)\int_0^1 \sin\frac{2\pi x}{a}\left\{-\frac{\lambda}{2}\left(\frac{\pi x}{a}\right) - \frac{\lambda^3}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a} + \frac{a_1}{2}\sqrt{2}\lambda^2\sinh\frac{\sqrt{2}\lambda\pi x}{a}\right\}d\left(\frac{x}{a}\right)$$

$$+\frac{1}{2}\int_0^1\left\{-\frac{\lambda}{2}\left(\frac{\pi x}{a}\right) - \frac{\lambda^3}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a} + \frac{a_1}{2}\sqrt{2}\lambda^2\sinh\frac{\sqrt{2}\lambda\pi x}{a}\right\}^2 d\left(\frac{x}{a}\right)\Bigg]$$

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$$\frac{1}{E^2ab}\int_0^a\int_0^b \tau_{xy}^2\,dx\,dy = \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\left[\frac{\lambda^2(1+\lambda^2)^2}{4(1+2\lambda^2)^4} + \frac{\lambda^4(1+\lambda^2)}{16(1+2\lambda^2)^3}(f\pi^2) + \frac{\lambda^6}{256(1+2\lambda^2)^2}(f\pi^2)^2\right.$$

$$+ \frac{\lambda^6}{4096(2+\lambda^2)^2}(f\pi^2)^2 + \frac{\lambda^6}{256(1+8\lambda^2)^2}(f\pi^2)^2 + \frac{\lambda^6}{16384(1+2\lambda^2)^2}(f\pi^2)^2$$

$$+\left\{\frac{\lambda(1+\lambda^2)}{(1+2\lambda^2)^2} + \frac{\lambda^3}{1+2\lambda^2}\frac{f\pi^2}{8}\right\}\left\{-\frac{\lambda}{2} + \frac{1}{4}\frac{\lambda^3}{1+\lambda^2} + \frac{1}{2}\frac{\lambda}{(1+2\lambda^2)^2}\right\}$$

$$+ \frac{\lambda^3}{32(2+\lambda^2)}(f\pi^2)\left\{+\frac{\lambda}{4} - \frac{2}{3}\frac{\lambda^3}{1+2\lambda^2} - \frac{1}{2}\frac{\lambda}{(1+2\lambda^2)(2+\lambda^2)}\right\}$$

$$+\frac{1}{2}\left\{\frac{\lambda^2}{4}\frac{\pi^2}{3} - \frac{2\lambda^4}{1+2\lambda^2} - \frac{1}{4}\frac{1}{(1+2\lambda^2)}\frac{1}{\sinh\sqrt{2}\lambda\pi}\left(\sqrt{2}\lambda\pi\cosh\sqrt{2}\lambda\pi - \sinh\sqrt{2}\lambda\pi\right)\right.$$

$$+ \frac{\lambda^6}{(1+2\lambda^2)^2}\left(\frac{\pi^2}{6}+\frac{1}{4}\right) - \frac{\lambda^4\pi}{(1+2\lambda^2)^2\sinh\sqrt{2}\lambda\pi}\left(-\frac{\sqrt{2}\lambda\cosh\sqrt{2}\lambda\pi}{1+2\lambda^2} + \frac{1}{\pi}\frac{(2\lambda^2-1)\sinh\sqrt{2}\lambda\pi}{(1+2\lambda^2)^2}\right)$$

$$\left.\left. + \frac{\pi^2\lambda^2}{4(1+2\lambda^2)^2\sinh^2\sqrt{2}\lambda\pi}\left(\frac{\sinh 2\sqrt{2}\lambda\pi}{4\sqrt{2}\lambda\pi} - \frac{1}{2}\right)\right\}\right]$$

$$\frac{E_1}{Eabt} = [illegible] \left[H_1(\lambda)(f\pi^2)^2 - H_2(\lambda)(f\pi^2) + H_3(\lambda) \right] + \frac{1}{2}\left(\frac{\sigma}{E}\right)^2$$

$$H_1(\lambda) = \frac{105}{32768} + \frac{17\lambda^4}{16384} + \frac{\lambda^4}{2048(1+8\lambda^2)^2} + \frac{\lambda^4}{32768(1+2\lambda^2)^2} + \frac{\lambda^4}{512(1+2\lambda^2)^2} + \frac{\lambda^4}{2048(2+\lambda^2)^2}$$

$$+ \frac{\lambda^6}{256(1+2\lambda^2)^2} + \frac{\lambda^6}{4096(2+\lambda^2)^2} + \frac{\lambda^6}{256(1+8\lambda^2)^2} + \frac{\lambda^6}{16384(1+2\lambda^2)^2}$$

$$= \frac{105}{32768} + \frac{17\lambda^4}{16384} + \frac{\lambda^4}{2048(1+8\lambda^2)} + \frac{\lambda^4}{32768(1+2\lambda^2)} + \frac{\lambda^4}{512(1+2\lambda^2)} + \frac{\lambda^4}{4096(2+\lambda^2)}$$

$$\boxed{H_1 = \frac{105}{32768} + \frac{17\lambda^4}{16384} + \frac{\lambda^4}{2048(1+8\lambda^2)} + \frac{65}{32768}\frac{\lambda^4}{(1+2\lambda^2)} + \frac{1}{4096}\frac{\lambda^4}{(2+\lambda^2)}}$$

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$$H_2(\lambda) = \frac{5}{48} + \frac{1}{32}\frac{\lambda^4}{(1+2\lambda^2)^3} + \frac{1}{64}\frac{\lambda^4}{(1+2\lambda^2)^2} + \frac{1}{16}\frac{\lambda^4}{(1+2\lambda^2)^3} + \frac{1}{96}\frac{\lambda^4}{(2+\lambda^2)(1+2\lambda^2)}$$

$$- \frac{1}{32}\frac{\lambda^4}{(1+2\lambda^2)(2+\lambda^2)^2} \quad \frac{1}{16}\frac{\lambda^4(1+\lambda^2)}{(1+2\lambda^2)^3} + \frac{1}{16}\frac{\lambda^4}{(1+2\lambda^2)} - \frac{1}{32}\frac{\lambda^6}{(1+2\lambda^2)^2} - \frac{1}{16}\frac{\lambda^4}{(1+2\lambda^2)^3}$$

$$- \frac{1}{128}\frac{\lambda^4}{(2+\lambda^2)} + \frac{1}{48}\frac{\lambda^6}{(2+\lambda^2)(1+2\lambda^2)} + \frac{1}{64}\frac{\lambda^4}{(1+2\lambda^2)(2+\lambda^2)^2}$$

$$\boxed{H_2 = \frac{5}{48} + \frac{3}{64}\frac{\lambda^4}{(1+2\lambda^2)} + \frac{\lambda^4}{384(2+\lambda^2)} - \frac{1}{64}\frac{\lambda^4}{(1+2\lambda^2)(2+\lambda^2)^2}}$$

$$H_3(\lambda) = \cancel{\frac{\pi^2}{8}} - \cancel{\frac{3}{16}}\Big] + \cancel{\frac{1}{16}} + \cancel{\frac{1}{8}\frac{\lambda^4}{(1+2\lambda^2)^4}} - \cancel{\frac{1}{4}\frac{\lambda^2}{(1+2\lambda^2)}} - \cancel{\frac{1}{8}\frac{1}{(1+2\lambda^2)}} + \cancel{\frac{1}{8}\frac{\lambda^4}{(1+2\lambda^2)^3}} + \cancel{\frac{1}{2}\frac{\lambda^4}{(1+2\lambda^2)^4}}$$

$$+ \cancel{\frac{1}{4}\left(\frac{\pi^2}{6} - \frac{1}{4}\right)\left(\frac{\lambda^2}{1+2\lambda^2}\right)^2} + \cancel{\frac{1}{4}\frac{\lambda^2}{(1+2\lambda^2)^3}\frac{\cosh\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)}} - \cancel{\frac{\lambda^4}{(1+2\lambda^2)^4}}$$

$$+ \frac{1}{16(1+2\lambda^2)^2}\frac{\frac{1}{2} + \frac{1}{4\sqrt{2}\lambda\pi}\sinh 2\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)^2}\Bigg] + \cancel{\frac{\lambda^2(1+\lambda^2)^2}{4(1+2\lambda^2)^4}} - \cancel{\frac{1}{2}\frac{\lambda^2(1+\lambda^2)}{(1+2\lambda^2)^2}} + \cancel{\frac{1}{4}\frac{\lambda^4(1+\lambda^2)}{(1+2\lambda^2)^3}}$$

$$+ \cancel{\frac{1}{2}\frac{\lambda^2(1+\lambda^2)}{(1+2\lambda^2)^4}} + \cancel{\frac{\lambda^2}{8}\frac{\pi^2}{3}} - \cancel{\frac{\lambda^4}{(1+2\lambda^2)}} - \cancel{\frac{1}{8}\frac{1}{(1+2\lambda^2)}\frac{\cosh\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)}} + \cancel{\frac{1}{8}\frac{1}{(1+2\lambda^2)}} + \cancel{\frac{\lambda^6}{2(1+2\lambda^2)^2}\left(\frac{\pi^2}{6} + \frac{1}{4}\right)}$$

$$+ \cancel{\frac{1}{2}\frac{\lambda^4}{(1+2\lambda^2)^3}\frac{\cosh\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)}} + \cancel{\frac{1}{2}\frac{\lambda^4(1-2\lambda^2)}{(1+2\lambda^2)^4}} + \cancel{\frac{1}{16(1+2\lambda^2)^2}\frac{\frac{1}{2} + \frac{1}{4\sqrt{2}\lambda\pi}\sinh 2\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)^2}}$$

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$$H_3(\lambda) = -\frac{1}{16(1+2\lambda^2)^2}\,\frac{\cosh\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)} + \frac{\pi^2}{8} - \frac{1}{8} + \cancel{\frac{1}{4}\left(\frac{\pi^2}{6}-\frac{1}{4}\right)\left(\frac{\lambda^2}{1+2\lambda^2}\right)^2} + \cancel{\frac{1}{2}\left(\frac{\pi^2}{6}+\frac{1}{4}\right)\frac{\lambda^6}{(1+2\lambda^2)^2}}$$

$$+ \frac{3}{8}\,\frac{\lambda^2(2-\lambda^2)}{(1+2\lambda^2)^3} - \frac{1}{8} + \frac{1}{8}\,\frac{\lambda^4(3+2\lambda^2)}{(1+2\lambda^2)^3} - \frac{1}{2}\,\frac{\lambda^2(1+\lambda^2)}{(1+2\lambda^2)^2} + \cancel{\frac{\lambda^2}{24}}\pi^2 + \frac{1}{8}\,\frac{(1-8\lambda^4)}{(1+2\lambda^2)}$$

$$= -\frac{1}{16(1+2\lambda^2)^2}\,\frac{\cosh\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)} + \frac{\pi^2}{8} - \frac{1}{4} + \frac{\pi^2}{24}\,\frac{\lambda^4}{(1+2\lambda^2)} + \frac{\pi^2}{24}\lambda^2 - \frac{1}{16}\,\frac{\lambda^4(1-2\lambda^2)}{(1+2\lambda^2)^2}$$

$$+ \frac{1}{4}\,\frac{\lambda^2(3+\lambda^4)}{(1+2\lambda^2)^3} - \frac{1}{2}\,\frac{\lambda^2(1+\lambda^2)}{(1+2\lambda^2)^2} + \frac{1}{8} - \frac{1}{4}\,\frac{\lambda^2(1+4\lambda^2)}{(1+2\lambda^2)}$$

$$= -\frac{1}{16(1+2\lambda^2)^2}\,\frac{\cosh\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)} + \frac{\pi^2}{8}\left\{1 + \frac{1}{3}\,\frac{\lambda^2(1+3\lambda^2)}{1+2\lambda^2}\right\} - \frac{1}{8} - \frac{1}{16}\,\frac{\lambda^4}{(1+2\lambda^2)}$$

$$+ \frac{1}{4}\,\frac{\lambda^6}{(1+2\lambda^2)^2} - \frac{1}{2}\,\frac{\lambda^2(1+\lambda^2)}{(1+2\lambda^2)^2} + \frac{1}{4}\,\frac{\lambda^2(3+\lambda^4)}{(1+2\lambda^2)^3} - \frac{1}{4}\,\frac{\lambda^2(1+4\lambda^2)}{(1+2\lambda^2)}$$

$$H_3 = \frac{1}{8}(\pi^2-1) - \frac{1}{16(1+2\lambda^2)^2}\,\frac{\cosh\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)} + \frac{\pi^2}{24}\,\frac{\lambda^2(1+3\lambda^2)}{(1+2\lambda^2)} - \frac{1}{16}\,\frac{\lambda^2(4+17\lambda^2)}{(1+2\lambda^2)} + \frac{\lambda^2(\lambda^4-2\lambda^2-2)}{4(1+2\lambda^2)^2}$$

$$+ \frac{1}{4}\,\frac{\lambda^2(3+\lambda^4)}{(1+2\lambda^2)^3}$$

4gl

$$\frac{P}{abtE} = \left(\frac{a}{R}\right)^4 \left(\frac{f}{\delta}\right)^2 \left[H_1 (1-\pi^2)^2 - H_2 (1-\pi^2) + H_3 \right] + \left(\frac{t}{R}\right)^2 \left(\frac{f}{\delta}\right)^2 \pi^4 \left[\frac{1}{32}(1+\lambda^4) + \frac{1}{48}\lambda^2 \right]$$

$$- \left(\frac{a}{R}\right)^2 \left(\frac{f}{\delta}\right)^2 \left(\frac{3}{8}\lambda^2\pi^2\right) \frac{\sigma}{E} - \frac{1}{2}\left(\frac{\sigma}{E}\right)^2$$

$$\frac{3}{4}\lambda^2\pi^2\frac{\sigma}{E} = \left(\frac{a}{R}\right)^2 \left[4H_1 (1-\pi^2)^2 - 3H_2 (1-\pi^2) + 2H_3 \right] + \left(\frac{t}{R}\right)^2 \pi^4 \left[\frac{1}{16}(1+\lambda^4) + \frac{1}{24}\lambda^2 \right]$$

$$\lambda^2 K = \gamma^2 \left[\frac{16}{3} H_1 f^2 \pi^2 - 4 H_2 f + \frac{8}{3} H_3 \frac{1}{\pi^2} \right] + \frac{\pi^2}{\gamma^2} \left[\frac{1}{12}(1+\lambda^4) + \frac{1}{18}\lambda^2 \right]$$

$$= \frac{\pi^2}{\gamma^2} \left[\frac{64}{3} H_1 \left(\frac{\delta}{t}\right)^2 + \left\{ \frac{1}{12}(1+\lambda^4) + \frac{1}{18}\lambda^2 \right\} \right] + \frac{8}{3}\frac{\gamma^2}{\pi^2} H_3 - 8H_2\left(\frac{\delta}{t}\right)$$

$$= 2 \left[\frac{512}{9} H_1 H_3 \left(\frac{\delta}{t}\right)^2 + H_3 \left\{ \frac{2}{9}(1+\lambda^4) + \frac{4}{27}\lambda^2 \right\} \right]^{\frac{1}{2}} - 8H_2\left(\frac{\delta}{t}\right)$$

$$\lambda = 1.$$

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$$\frac{512}{9} H_1 = \frac{\cancel{105}^{35}}{64 \times \cancel{9}_3} + \frac{17}{32 \times 9} + \frac{1}{4 \times 81} + \frac{65}{64 \times 27} + \frac{1}{8 \times 27}$$

$$= 0.1822917 + 0.0590278 + 0.0030864 + 0.0376157 + 0.0046296$$

$$= 0.2866512$$

$$8 H_2 = \frac{40}{48} + \frac{3}{8}\frac{1}{3} + \frac{1}{48}\frac{1}{3} - \frac{1}{8}\frac{1}{27}$$

$$= 0.8333333 + 0.1250000 + 0.00694444 - 0.0046296$$

$$= 0.9606481$$

$$H_3 = 1.1087006 - \frac{4.444113}{16 \times 9} + \frac{\pi^2}{\cancel{24}_6}\frac{\cancel{4}}{3} - \frac{1}{16}\frac{21}{3} - \frac{1}{4}\frac{3}{9} + \frac{1}{27}$$

$$= 1.1087006 - 0.0308619 + 0.5498311 - 0.4375 - 0.0833333 + 0.0370370$$

$$= 1.1438735$$

$$K = 2\left[0.3278927 \left(\tfrac{d}{t}\right)^2 + 1.6778509\right]^{\frac{1}{2}} - 0.9606481 \left(\tfrac{d}{t}\right)$$

$$0.6557854 \left(\tfrac{d}{t}\right) = 0.9606481 \left[0.3278927 \left(\tfrac{d}{t}\right)^2 + 0.6778509\right]^{\frac{1}{2}}$$

$$0.4300545 \left(\tfrac{d}{t}\right)^2 = 0.6255512$$

$$\begin{array}{r} 0.3025940 \\ \hline 0.1274505 \end{array}$$

$$\left(\tfrac{d}{t}\right)^2 = 4.908189 \qquad \left(\tfrac{d}{t}\right) = 2.215444$$

$$K_{min} = 0.8965 \qquad K_0 = 1.646636$$

$\lambda = 0.5$ 485

$$\frac{512}{9} H_1 = 0.1822917 + \frac{17 \times 0.0625}{288} + \frac{0.0625}{108} + \frac{65}{\;} \frac{0.0625}{864} + \frac{1}{\;} \frac{0.0625}{162}$$

$$= 0.1822917 + 0.0625 \left(0.05932777 + 0.00925926 + 0.07523148 + 0.00617284 \right)$$

$$= 0.1822917 + 0.0093557 = 0.1916474 \qquad 0.7665896$$

$$8H_2 = 0.8333333 + 0.0625 \left(\frac{3}{8} \frac{1}{1.5} + \frac{1}{48} \frac{1}{2.25} - \frac{1}{8} \frac{1}{1.5 \times 2.25^2} \right)$$

$$= 0.8333333 + 0.0625 \left(0.25 + 0.00925926 - 0.01646091 \right)$$

$$= 0.8485082 \qquad 3.3940328$$

$$H_3 = 1.1087006 - \frac{1}{36} 2.274322 + \pi^2 \frac{0.25 \times 1.75}{36} - \frac{1}{\;} \frac{0.25 \times 8.25}{24}$$

$$- \frac{0.25 \times 2.4375}{9} + \frac{0.25 \times 3.0625}{13.5}$$

$$= 1.1087006 - 0.06317511 + 0.1199431 - 0.08593750 - 0.06770833 + 0.05671296$$

$$= 1.1476353 \qquad 4.5905412$$

$$K = 2 \left[3.5190611 \left(\frac{f}{t}\right)^2 + 5.0155909 \right]^{\frac{1}{2}} - 3.3940328 \left(\frac{f}{t}\right)$$

$$49.5351641 \left(\frac{f}{t}\right)^2 = 57.7762917 \qquad \left(\frac{f}{t}\right) = 2.534056$$

$$\begin{array}{r} 40.5376787 \\ \hline 8.9974854 \end{array} \qquad 6.421448$$

$$K_{min} = 1.4 - -$$